Physics Aptitude Test (PAT) Unofficial Solutions: 2018

University of Oxford Admissions Test

Physics, Engineering, Materials Science

Solutions by PhysOx Initiative
2020

Contents

P.I

PhysOx

1 Foreword

The solutions provided here are by no means an official set of answers; our aim was to provide unofficial solutions in order to provide those studying for the PAT with the means to check their answers as they attempt past paper questions. There is no mark breakdown but we aimed to provide detailed explanations on how to solve the questions and develop a good intuition for them. These solutions have been compiled by our team of Oxford Physics graduates who have all taken the PAT and our team members also have experience in marking the PAT and/or running the PAT summer programmes coordinated by the University of Oxford.

Some general tips for questions:

1) Try and **keep your calculations in terms of symbols and letters** until you see your calculations simplifying considerably by substituting numbers. This will genuinely reduce errors in your work and make it way easier for the marker to understand your thought process.

2) Give your **variables reasonable names**, for example don't call your initial velocity something like v_u and final velocity something like v_v, as you will definitely confuse the life out of everyone looking at your work not to mention yourself.

3) Your teachers may say this a lot and many of you probably ignore it but **drawing diagrams really helps!** Sometimes the best way to deal with paragraphs of information is a simple drawing which has all the important bits - it will also save you a lot of time!

4) Always **show your working** so it does not look like you picked an answer out of thin air especially for the longer answer questions so it's easier to pick up on exactly where you went wrong if you do go wrong. Fair enough you may have had a moment of next level inspiration but a couple of lines (or words here and there) just outlining your way of thinking really helps. That said, it doesn't have to be an essay!

5) Don't feel the need to rush, **relax** yourself and approach the questions. If you get really stuck on one, don't get too put off, just skip it and come back to it later if you get time.

2 Section A: Multiple Choice Questions

2.1 Answers

1	C
2	E
3	E
4	C
5	C
6	B
7	A
8	A
9	B
10	E
11	C
12	C

Table 1: Multiple Choice Answers.

2.2 Answers Explained

2.2.1 Qu1: Sequences and Series

This question is a nice ice-breaker. If a series doesn't immediately look like a simple geometric or arithmetic one or one of those Fibonacci-type ones involving sums, it is worth trying to look at patterns in the differences between terms. Also (though this is not relevant here) if you have a series with decimals with no obvious pattern, try and convert it to fractions first - usually the pattern is more likely to jump out in this form.

Now back to the question, we will look at the differences:

Figure 1: Qu1

Given the pattern in the differences, i.e. the sum of the previous two differences is the next difference, the next term is given by a difference of 14, then the next term should $23 + 14 = 37$.

The answer is C.

2.2.2 Qu2: Orbits

Orbits that we know of can be circular (like 2 and 3), elliptical (like 1), parabolic (like 4) or hyperbolic (like 5). Now seeing as spotting weirdly-shaped orbits is not the distinguishing factor, we need to look more closely at each of the orbits. First we must note that all systems must obey the conservation laws; the one which will help us understand which orbit is the imposter is conservation of angular momentum.

Angular momentum is defined as $L = rpsin\theta$, where r is the distance between the orbiter (let's call B) and the "orbitee" (let's call A) and p is the momentum. You could also write this as $L = mv_{perp}r$. Now if we look at the expression for L, m is a constant and $v_{perp}r$, which has dimensions $[L]^2/[T]$, could potentially vary. Length squared $[L]^2$ is an area and for L to be conserved we want $dL/dt = 0$ so this "area per unit time"-like quantity ($v_{perp}r$) must be a constant for L/m to also me constant. This is where Kepler's 2nd Law comes from: a planet will sweep out equal areas at equal times - though the way we justified it is very hand-wavey!

Now looking at our orbits we find we can justify all of them through this area consideration apart from 3. In principle, circular orbits are fine but we need A to be central otherwise to compensate for the area rule (as radius is varying) we would need to demand a change of speed in B. Now consider the situation of B moving around a perfectly circular orbit but with its speed varying. Circular motion relies on a constant centripetal force and say if radius decreases as velocity increases (for L conservation the factor of decrease/increase has to be the same i.e. $vr = constant$) then the centripetal force is forced to change: this is contradictory and not feasible!

The answer is then very clearly going to be E because it's weird.

2.2.3 Qu3: Dimensional Analysis

This one's a fun one if you spot it. The most familiar unit of the options given is C which you should immediately recognise as the units for force. Let's check if we can convince ourselves the others are also forces.

A. Coulombs is used for charge Q and Vm^{-1} for electric field (voltage divided by distance between plates). This is $QE = F$. So force does seem to have been a good lead and the answer is not A or C!

B. A is current I, T is magnetic field B and m is a length - so we have $BIl = F$ for the force due to a current carrying wire of length l. It's not B either!

D. J for work done or energy and m is a length so we have $WD/s = F$. Also not D, so hopefully the answer is E but let's check!

E. We have charge Q and a velocity v - most definitely not a force so E is the odd one out.

Answer: E.

2.2.4 Qu4: Probabilities

This is a binomial distribution:

$$X \sim B(3, 1/3)$$

The probability of going L is 1/3 and R is then 2/3. Let's use combinations to work through the options:

A. ${}^3C_0(1/3)^3 = 1/27$.

B. ${}^3C_1(1/3)^2(2/3) = 6/27$.

C. ${}^3C_2(2/3)^2(1/3) = 12/27$.

D. ${}^3C_3(2/3)^3 = 8/27$.

So C is most probable. For those who prefer a more visual approach, you can use a tree diagram too if that's more comfortable.

2.2.5 Qu5: Geometric Series

The n^{th} cup has a fraction α^{n-1} of the original. We want the sum to infinity to be less than 3 (apply a the sum to infinity formula for a geometric series with ratio $r = \alpha$ and first term $a = 1$:

$$\frac{a}{1-r} < 3$$

$$\frac{1}{1-\alpha} < 3 \implies \alpha < 1 - 1/3$$

$$\alpha < 2/3$$

The answer is C.

2.2.6 Qu6: Harmonics

The center and ends are nodes. Let's draw a diagram.

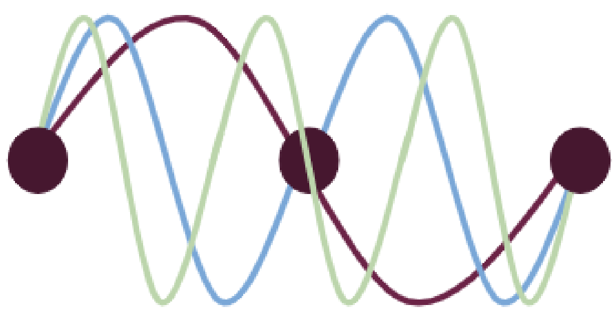

Figure 2: Qu6

Only any integer multiples of λ can fit in the space so $m\lambda_m = L$, for any integer value of m: B.

2.2.7 Qu7: Work Done

The car must lose all of its kinetic energy before it gets to the cat. Therefore, all of the work done by the brakes to stop the car should be equal to the initial kinetic energy of the car (which must be lost).

$$WD = Fd = -\frac{1}{2}mv^2$$

$$F = \frac{-mu^2}{2d}$$

The answer is A.

2.2.8 Qu8: Stationary points

You can expand this one and differentiate or use product rule but this one is a pretty gentle stationary points question. We find the point at which $\frac{dy}{dx} = 0$.

$$y = x^2 - 2x - 3$$

$$\frac{dy}{dx} = 2x - 2 = 0 \implies x = 1$$

OR you can even notice that you have a quadratic, a function which is symmetric and so the x of its minimum point will be the midpoint of the intercepts (which you can read off as it's already in factorised form):

$$x = \frac{3 + (-1)}{2} = 1$$

Whichever method you choose, you get A.

2.2.9 Qu9: Normals

The gradients $m_{1,2}$ of orthogonal (or perpendicular) lines follow the relation:

$$m_1 m_2 = -1$$

The gradient of the given line is 2, therefore the line we want will have gradient $m_2 = -1/2$.

Now we need to know a point this line will run through and we are given that the two lines intersect when $x = 1$. At this point, from the first line equation, the y value is 0 and so the point it runs through is $(1, 0)$. Now we can find the equation of the perpendicular line:

$$y = \frac{-1}{2}x + c \implies 0 = -1/2 + c$$

$$\therefore y = -\frac{1}{2}x + \frac{1}{2}$$

The answer is B.

2.2.10 Qu10: Inequalities

This cubic needs factorising. The constant term is 1 so chances are it will be some combination of ± 1s making up the roots and there needs to be two negatives and a positive as otherwise the constant term wouldn't be positive: $(x-1)(x-1)(x+1) \geq 0$ seems like a good guess and you could check it! Of course feel free to use the factor theorem and/or long division if that makes you feel more comfortable - should give you the same answer.

We have a repeated root at $x = 1$ and a "normal" root at $x = -1$ so the graph only touches $x = 1$ and goes through $x = -1$. A quick sketch reveals all!

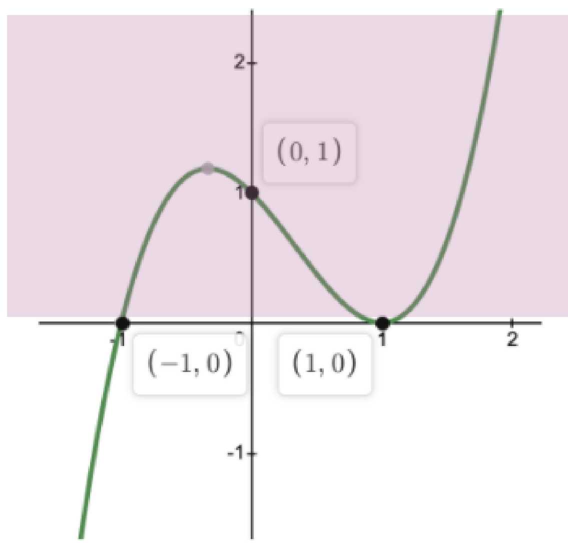

Figure 3: Qu10

The graph is positive for all values of y when $x \geq -1$. The answer is E.

2.2.11 Qu11: Stress

The wall will support the roof as long as its maximum stress is greater than the stress "inflicted on it" by the roof. If we find the stress exerted by the roof we then know that this is the minimum stress our ideal wall-material candidate must be able to handle. Stress is just force per unit cross-sectional area:

$$Stress = \frac{F}{A}$$

Write down what we know concisely:

- mass per unit area $100kgm^{-2}$
- $50m \times 100m$, $0.1m$ thick

So the mass of the roof is

$$(100 \times 50)100 = 500000kg$$

and this is supported by the walls, which have a cross-sectional area

$$(100 \times 50) - (99.8 \times 49.8) = 29.96m^2$$

Note we will need this in mm^2 later. This came from the fact the walls have a $0.1m$ thickness so you can take away the purple area from the green area.

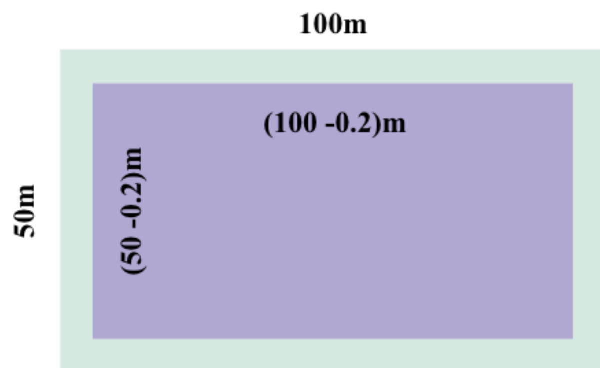

Figure 4: Qu11

So putting it all together:

$$S_{min} = \frac{5 \times 10^5 g}{29.96 \times 10^6} = 0.17Nmm^{-2}$$

So any material with $S_{max} \geq S_{min}$ is fine. Looking at the options, all of them are and so the answer is C!

2.2.12 Qu12: Refraction

The refractive index increases by a fraction each time so let's say this:

$$n_{i+1} > n_i$$

From Snell's Law:

$$n_i sin\theta_i = n_{i+1} sin\theta_{i+1}$$

$$sin\theta_{i+1} = \frac{n_i}{n_{i+1}} sin\theta_i$$

Now let's consider the next boundary

$$sin\theta_{i+2} = \frac{n_{i+1}}{n_{i+2}} \frac{n_i}{n_{i+1}} sin\theta_i$$

We get cancellations! Therefore, we can expect for the λ^{th} term:

$$sin\theta_{i+\lambda} = \frac{n_i}{n_{i+\lambda}}$$

And as the right side gets smaller and smaller due to later refractive indices getting progressively greater then,

$$sin\theta_{i+\lambda} \longrightarrow 0 \implies \theta_\infty \longrightarrow 0$$

Answer is C.

3 Section B: Long Answer Questions

3.0.1 Qu13: Capacitor Networks

Capacitors in series and parallel works the opposite way to resistors in series and parallel but they already tell you that. Let's sort out each branch first and then combine.

Branch 1

$$\frac{1}{C_1} = \frac{1}{C} + \frac{1}{C} \implies C_1 = C/2$$

Branch 2

$$\frac{1}{C_2} = \frac{1}{C} + \frac{1}{2C} \implies C_2 = 2C/3$$

Branch 3

$$C_3 = 3C$$

Therefore overall:

$$C_T = 3C + C/2 + 2C/3 = \frac{25C}{6}$$

3.0.2 Qu14: Logarithms and Exponentials

There are other ways to do this but perhaps the method which assumes minimal "specific" knowledge of any rules is as follows.

$$log_x 25 = log_5 x$$

Set $log_x 25 = a$ and $log_5 x = a$, the first of which suggests that

1. i) $x^a = 25$

and the second which gives us the following:

$$a = log_5 x$$

1. ii) $5^a = x$

Now applying the definition for x in ii) in place of the x in i).

$$(5^a)^a = 25 \implies 5^{a^2} = 25$$

$$a = \pm 2$$

Then from ii).

$$x = 5^a = 5^{\pm 2}$$

3.0.3 Qu15: A Car in Motion

a) Let's write down what we know.

1. Starting at $t = t_0$, speed $v = v_1$ maintained for time Δt_1.

2. Starting at $t = t_1$, speed $v = v_2$ maintained for time Δt_2.

3. $v_2 > v_1$ and $\Delta t_2 < \Delta t_1$.

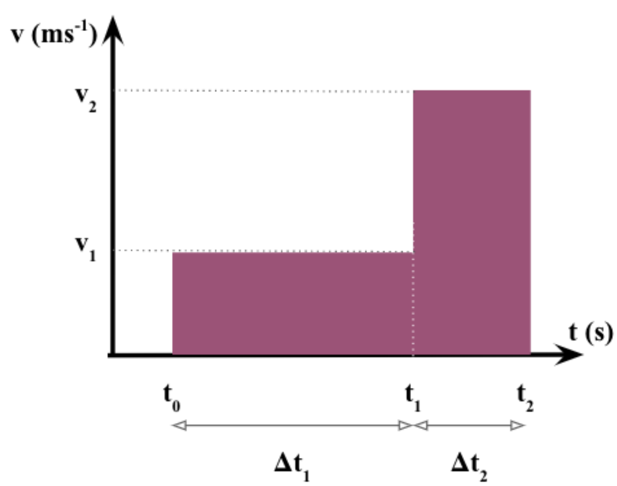

Figure 5: Qu15a

b) Finding the time-averaged speed.

$$\langle v \rangle_t = \frac{v_1 \Delta t_1 + v_2 \Delta t_2}{\Delta t_1 + \Delta t_2} = \frac{1(2) + 2(1)}{2+1} = \frac{4}{3} ms^{-1}$$

c) Graph of $v - s$.

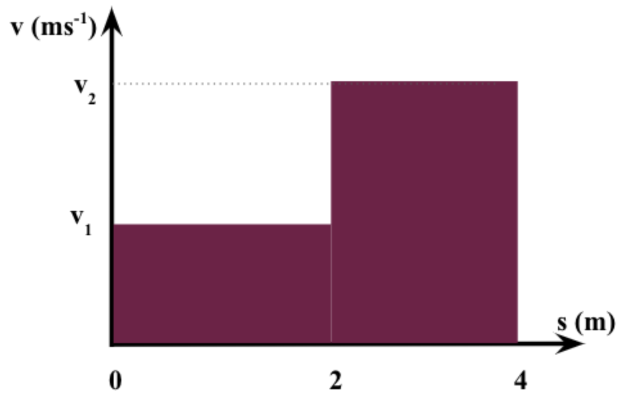

Figure 6: Qu15c

d) Distance-weighted average speed.

$$\langle v \rangle_s = \frac{v_1 \Delta s_1 + v_2 \Delta s_2}{\Delta s_1 + \Delta s_2} = \frac{1(2) + 2(2)}{2+2} = \frac{3}{2} ms^{-1}$$

e) The conventional definition of average speed is a time-averaged speed.

$$\langle v \rangle_t = \langle v \rangle_c$$

f) To find the average speed, can integrate $v(t)$ and normalise by the total time t.

$$\langle v \rangle_c = \frac{1}{T} \int_0^T v(t) dt$$

3.0.4 Qu16: Geometry

Let's find the areas and equate!

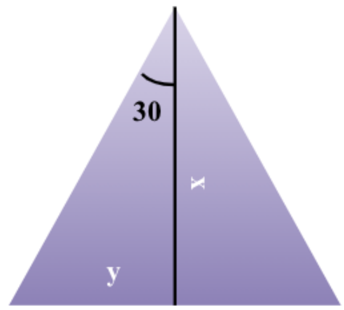

Figure 7: Qu16

1. Area of circle: πr^2.

2. Area of hexagon:

Looking at the triangle, $tan30 = y/x \implies y = \frac{x}{\sqrt{3}}$. The total area of the hexagon is $6xy = \frac{2\times 3x^2}{\sqrt{3}} = 2\sqrt{3}x^2$.

We are told that the black area is a quarter of the hexagon area:

$$2\sqrt{3}x^2 = 4\pi r^2$$

$$x = (\frac{2\sqrt{3}}{3}\pi)^{1/2}r$$

3.0.5 Qu17: Taylor expansions and Integrals

Let's do the integral "normally":

$$\int_0^{0.1}(1+x)^9 dx = [\frac{(1+x)^{10}}{10}]_0^{0.1} = 0.15937...$$

If the result is to be better that 10% then we should probably figure out the 10% lower and upper bound.

10% Upper bound: ~ 0.1753
10% Lower bound: ~ 0.1434
Now consider the series expansion and the integral of that:

$$\int_0^{0.1}[1 + 9x + {}^9C_2x^2 + {}^9C_3x^3 + ...]dx$$

Let's expand to fourth order because typically this order is usually (more than) enough.

$$= [x + \frac{9x^2}{2} + 12x^3 + ...]_0^{0.1}$$

$$= 0.1 + 0.045 + 0.012 + ...$$
$$= 0.145 + ...$$

This value is above the lower bound by the 2nd order sum so expanding to the x^{th} term does the trick!

3.0.6 Qu18: Forces in Equilibrium, Gravitational and Electric Forces

a) Consider the equilibrium of forces.

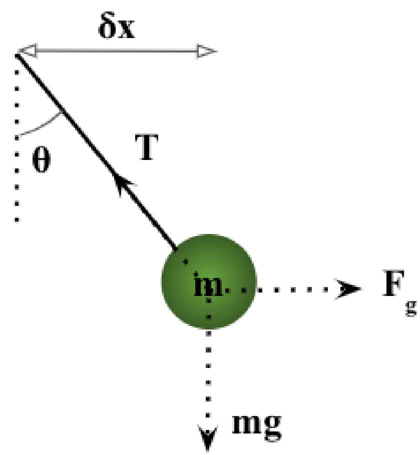

Figure 8: Qu18

In the vertical direction:

$$Tcos\theta = mg$$

In the horizontal direction:

$$Tsin\theta = F_g$$

Combining these two together and equating to the gravitational attraction (Newton's Law of Gravitation) between the two masses which are separated by a distance $x - 2\delta x$:

$$tan\theta = \frac{F_g}{mg} = \frac{Gm^2}{(x - \delta x)^2} \frac{1}{mg}$$

Now assuming that θ is small, we take the small angle approximation and from the given diagram can find that $tan\theta \simeq sin\theta \simeq \theta \simeq \frac{\delta x}{L}$. Thus combining with the lass expression we had incorporating F_g, we get:

$$\frac{\delta x}{L} = \frac{Gm}{g} \frac{1}{(x - 2\delta x)^2}$$

Here, we can almost work backwards a bit. The equation we are trying to show holds looks like it could come out of having to use the quadratic formula. So we need to expand the expression we have for δx to something which is quadratic.

Let's expand the denominator first.

$$\frac{\delta x}{L} = \frac{Gm}{g}\frac{1}{x^2}\frac{1}{(1 - 4\frac{\delta x}{x} + 4(\frac{\delta x}{x})^2)}$$

To get a quadratic we need to multiple the denominator over and also actually only consider the denominator to second order (the $(\frac{\delta x}{x})^2$ is not needed). This move is justified, and is probably our main assumption, as we can say that δx is assumed to be very small so the squared term is not going to offer any significant contribution in the solution.

$$\frac{\delta x}{L} = \frac{Gm}{g}\frac{1}{x^2}\frac{1}{(1 - 4\frac{\delta x}{x})}$$

$$g\frac{\delta x}{L}x^2(1 - 4\frac{\delta x}{x}) = Gm$$

$$\frac{x^2 g}{L}\delta x - \frac{4g}{L}\delta x^2 x - Gm = 0 \implies \delta x^2 - \frac{x}{4}\delta x - \frac{GmL}{gx} = 0$$

Using the quadratic formula:

$$\delta x \simeq \frac{x}{8} \pm \sqrt{(\frac{x}{8})^2 + \frac{LmG}{4xg}}$$

QED.

b) Like charges repel so the electrostatic or Coulomb force will need to counteract the gravitational attraction to get back to the vertical position:

$$\frac{Gmm}{r^2} = \frac{kQQ}{r^2} \implies Q = \sqrt{\frac{G}{k}}m$$

3.0.7 Qu19: Circuits

a) To find which is brighter when both bulbs are connected together, we must find the resistances. Here's what we know:

- $P_A = 100W = V_A^2/R_A = 100^2/R_A \implies R_A = 100\Omega$
- $P_B = 20W = 100^2/R_B \implies R_B = 500\Omega$

In the circuit given, the current going through both A and B will be the same (as it is a series circuit). We can use this fact and calculate the ratio of powers.

$$\frac{P_A}{P_B} = \frac{I^2 R_A}{I^2 R_B} = \frac{100}{500} = \frac{1}{5}$$

Therefore more power dissipated through B than A and hence B is brighter.

b) In parallel voltages are the same so we'll exploit this fact to find our power ratio. We like cancellations so we will use the power formula using V instead of I this time.

$$\frac{P_A}{P_B} = \frac{V_A^2}{R_A}\frac{R_B}{V_B^2} = \frac{500}{100} = 5$$

This time bulb A is brighter.

3.0.8 Qu20: Problem Solving given a Context: Crystal Planes

Lots of context in this question but actually if we strip this one down, it's a pretty nice question!

a) The three lattive types are described and we are told that (111), (200), (220) and (311) planes are found. This only fits the face centered cubic definition!

Answer: FCC

b) We are given an equation and some numbers. Essentially this part is testing our patience and ability to punch numbers into a calculator. First rearrange for a:

$$a = d\sqrt{h^2 + k^2 + l^2}$$

Then substitute noting that they have given us d and the planes hkl:

d (nm)	**a (nm)**
0.224	0.388
0.195	0.390
0.137	0.387
0.117	0.388

Figure 9: Qu20

The best estimate for a is probably the mean.

$$\langle a \rangle = \frac{\Sigma a}{4} \sim 0.388nm$$

c) Deformed object due to pressure.

The volume initially (as a cube) was $(Na)^3 = L^3$. Now the cube has become a cuboid but the volume is conserved. The other two dimensions can be defined by a length x:

$$(Na)^3 = (\frac{2}{3}Na)x^2$$

$$x^2 = \frac{3}{2}(Na)^2$$

$$x = (Na)\sqrt{\frac{3}{2}}$$

3.0.9 Qu21: Circular Motion

A good example of an "in-context" question which isn't actually as complicated as it may look initially. Don't get put off by massive chunks of text and weird diagrams! You'll find this question actually guides you through quite nicely.

a) To pull the child closer to the center, need to "overcome" the centripetal force.

$$F(r) = mr\omega^2$$

b) Work done by the child to reach the centre. They also technically tell you what to do so as long as you got part a) correct, you're good to go. If you didn't get a) then you can still get the marks for this part for correctly integrating your expression for $F(r)$.

$$WD = -\int_{r_0}^{0} mr\omega(r)^2 dr = \int_{0}^{r_0} mr\omega(r)^2 dr$$

c) Work done is converted to (rotational) kinetic energy of the child. Just based on part a), the angular speed will increase as the child approaches the center.

d) We are told the following:

$$J = I\omega = (mr^2 + I_p)\omega$$

$$\omega = \frac{J}{mr^2 + I_p}$$

e) Combining part b) and part d) we basically just have an integration problem now. Also note the hint tells us a lot about the kind of expression we should be integrating!

$$WD = \int_{0}^{r_0} \frac{mrJ^2}{(mr^2 + I_p)^2} dr = \int_{0}^{r_0} \frac{\frac{r}{m}J^2}{(r^2 + \frac{I_p}{m})^2} dr$$

We can use the integral they have given us in the hint to solve this.

$$= \frac{J^2}{m}[\frac{-1}{2(\frac{I_p}{m} + r^2)}]_0^{r_0}$$

$$= \frac{-J^2}{2m}\left(\frac{1}{\frac{I_p}{m}+r_0^2} - \frac{m}{I_p}\right)$$
$$\therefore WD = \frac{J^2}{2}\left(\frac{1}{I_p} - \frac{1}{I_p + mr_0^2}\right)$$

3.0.10 Qu22: Geometry

Let's factorise the circle equation using some good old completing the square.

$$x^2 - 8x + y^2 + 4y + 4 = 0$$
$$(x-4)^2 - 16 + (y-2)^2 - 4 + 4 = 0$$

Cancelling and rearranging:

$$(x-4)^2 + (y-2)^2 = 16$$

This is very clearly a circle with radius $r = 4$ and center $(4, -2)$. The area of the circle is therefore $A_o = 16\pi$. Now we can also make a quick plot of the graph/ area we are interested in.

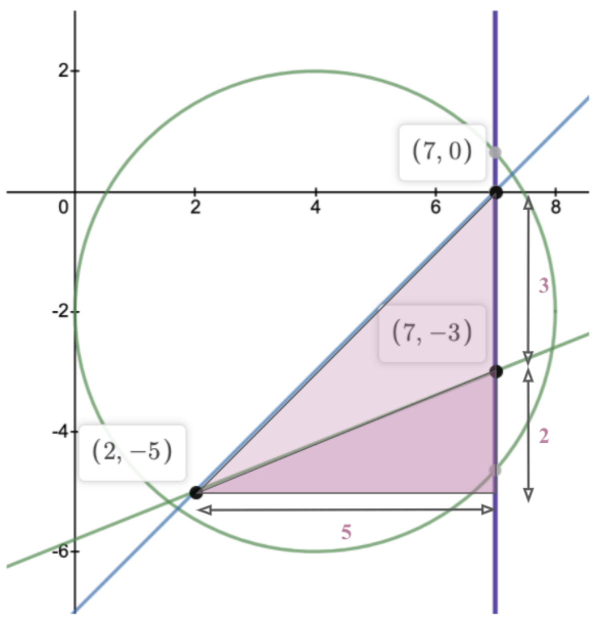

Figure 10: Qu22

To find the area of the triangle, we need to find the intersections and identify these on the plot to make our lives easier. Intersections:

$$x - 7 = \frac{1}{5}(2x - 29)$$

$$5x - 35 = 2x - 29$$
$$3x = 6$$
$$x = 2, y = -5 \implies (2, -5)$$

Also when $x = 7$, $y = 0$ so $(7, 0)$ and for $y = \frac{1}{5}(14 - 29) = -3$ so $(7, -3)$ is the last vertex of the triangle. From the diagram, the area of the triangle is:

$$A_\Delta = \frac{1}{2}[(5 \times 5) - (5 \times 2) = \frac{15}{2}$$

The area we want is then:

$$A = A_o - A_\Delta = 16\pi - \frac{15}{2}$$

3.0.11 Qu23: Functions and Domains

The function is

$$f(x) = \frac{\sqrt{x^2 - 2}}{ln(3x + 10)}$$

- Firstly, if we want this to be real we need $x^2 \geq 2$. This gives us two conditions:
$$x \leq -\sqrt{2}, x \geq \sqrt{2}$$
- The domain of the graph of $ln(a)$ is basically any value of $a > 0$. Thus:
$$3x + 10 > 0 \implies x > \frac{-10}{3}$$
- Now we also need to make sure $ln(3x + 10) \neq 0$ which will only happen when $3x + 10 = 1$. Then:
$$3x + 10 \neq 1 \implies x \neq -3$$

Combining these ranges:

$$-\frac{10}{3} < x < -3$$
$$-3 < x < -\sqrt{2}$$
$$x > \sqrt{2}$$

This set of inequalities should define the ranges for which the function is real and finite.

P.I

PhysOx