Physics Aptitude Test (PAT) Unofficial Solutions: 2020

University of Oxford Admissions Test

Physics, Engineering, Materials Science

Solutions by PhysOx Initiative
2020

Contents

P.I

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1 Foreword

The solutions provided here are by no means an official set of answers; our aim was to provide unofficial solutions in order to provide those studying for the PAT with the means to check their answers as they attempt past paper questions. There is no mark breakdown but we aimed to provide detailed explanations on how to solve the questions and develop a good intuition for them. These solutions have been compiled by our team of Oxford Physics graduates who have all taken the PAT and our team members also have experience in marking the PAT and/or running the PAT summer programmes coordinated by the University of Oxford.

Some general tips for questions:

1) Try and **keep your calculations in terms of symbols and letters** until you see your calculations simplifying considerably by substituting numbers. This will genuinely reduce errors in your work and make it way easier for the marker to understand your thought process.

2) Give your **variables reasonable names**, for example don't call your initial velocity something like v_u and final velocity something like v_v, as you will definitely confuse the life out of everyone looking at your work not to mention yourself.

3) Your teachers may say this a lot and many of you probably ignore it but **drawing diagrams really helps!** Sometimes the best way to deal with paragraphs of information is a simple drawing which has all the important bits - it will also save you a lot of time!

4) Always **show your working** so it does not look like you picked an answer out of thin air especially for the longer answer questions so it's easier to pick up on exactly where you went wrong if you do go wrong. Fair enough you may have had a moment of next level inspiration but a couple of lines (or words here and there) just outlining your way of thinking really helps. That said, it doesn't have to be an essay!

5) Don't feel the need to rush, **relax** yourself and approach the questions. If you get really stuck on one, don't get too put off, just skip it and come back to it later if you get time.

2 Section A: Multiple Choice Questions

2.1 Answers

1	C
2	B
3	D
4	B
5	D
6	C
7	A
8	D
9	B
10	A
11	D
12	E

Table 1: Multiple Choice Answers.

2.2 Answers Explained

2.2.1 Qu1: Atoms and Elements

Look at each statement by turn:

1. False: Proton number is 6 for both.
2. True: total number of nucleons is 15 for one and 14 for the other.
3. True: Oxygen has a nuclear charge of $8+$ while Nitrogen has $7+$.
4. True: $\frac{18}{8} > \frac{12}{6}$.
5. False: Nitrogen has $14 - 7 = 7$ neutrons and Carbon has $13 - 6 = 7$.

The answer is C.

2.2.2 Qu2: Transformations

There are two methods you can follow for this question:

1. The Graphical Method
2. The Matrix Method

Both methods are outlined below[1]!

The Graphical Method

As the name implies, we will plot the triangle, apply the transformations and then try to see which of our options best describe the overall mapping.

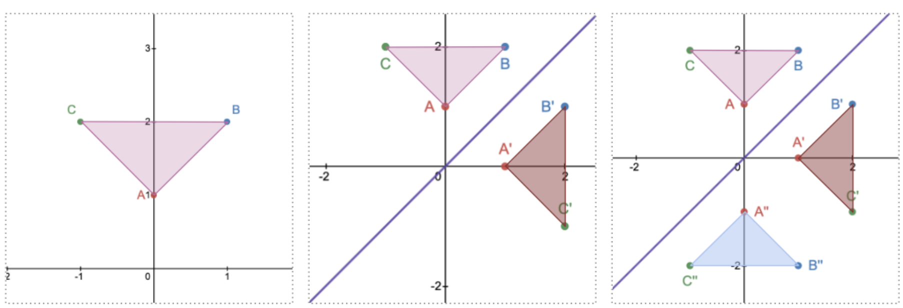

Figure 1: Qu2

[1]It is likely you may not have seen the matrix method before - don't be alarmed if this is the case! For those of you who may have covered this in your A Level Further Maths lessons, as you will see, it can be a pretty neat trick!

Comparing the pink and the blue triangles in Figure 1, it is clear that we have a reflection in the line $y = 0$. The answer is B!

The Matrix Method

Transformations (and also operators) in n-dimensional space can be very effectively described by matrices. We have two transformations in this case, each of which need a matrix description. These can be worked out (we will look at reflection as an example) but usually there are a few you can just learn off-by-heart so you don't have to figure it out each time!

Reflection in $y = x$

This is a transformation from $(x, y) \longrightarrow (y, x)$, i.e.:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \tag{1}$$

, where we need to find the coefficients a, b, c, d. Expanding Equation 1 to find appropriate values for the coefficients, we get $ax + by = y$ and $cx + dy = x$. This gives us $a = d = 0$ and $b = c = 1$. The $y = x$ reflection matrix is therefore:

$$T_{xyref} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Clockwise Rotations

Similarly the rotation matrices in various coordinate systems can be figured out with a bit of trig. We won't do this from scratch here[2] but the general matrix for clockwise rotations is given:

$$T_{clock} = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}. \tag{3}$$

Note that an anti-clockwise rotation is the equivalent of substituting $\theta \longrightarrow -\theta$ in the above. A 90° clockwise rotation is therefore:

$$T_{90c} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

Now for those who have studied matrices, you would know the order in which you multiply them makes a difference so we need to take extra case to apply the transformations in the correct order. We want to apply T_{xyref} then T_{90c} to (x, y); this is $(x'', y'') = T_{90c} T_{xyref} (x, y)^T$. The order appears reversed when we

[2] You can have a go yourself at trying to derive the general 2D rotation matrices following a similar method to the reflection case above. Think about how (x, y) coordinates relate to the rotation angle θ in your normal Cartesian system.

multiply!

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} \tag{5}$$

This is a transformation mapping $(x, y) \longrightarrow (x, -y)$: a reflection in $y = 0$ - B again!

This method may look long looking at the above but ultimately if you know what transformations to use, it is a one liner (Equation 5)!

2.2.3 Qu3: Circuits

The voltage across each branch is the same (rule of parallel circuits). The rules which we will use:

$$V = IR \tag{6}$$

So the higher the resistance, the lower the current across each branch.

$$R_{Tot}^{series} = \sum_i R_i \tag{7}$$

$$R_{Tot}^{para} = \sum_i \frac{1}{R_i} \tag{8}$$

Calculating the resistances across each branch first:

- Branch A

$$R + \frac{1}{\frac{1}{R} + \frac{1}{2R}} = \frac{5R}{3} \implies I_A = \frac{3V}{5R} \tag{9}$$

- Branch D

$$R + \frac{R}{3} = \frac{4R}{3} \implies I_D = \frac{3V}{4R} \tag{10}$$

- Branch E

$$\frac{7R}{5} \implies I_D = \frac{5V}{7R} \tag{11}$$

We can see that branch D has the highest current (lowest resistance). Paths B and C would consist of currents that are fractions of the total in branch A so we don't need to calculate these at all.

The answer is D!

2.2.4 Qu4: Logarithms and Exponentials

Note $log_2 x$ is written as $log x$ in the following.

$$log x + log(2x+3) = 1 \tag{12}$$

$$log x(2x+3) = 1 \tag{13}$$

Doing $2^{\cdots}$ to both sides to "undo" the log and rearranging:

$$2x^2 + 3x - 2 = 0 \tag{14}$$

$$2x^2 + 4x - x - 2 = 0 \implies 2x(x+2) - 1(x+2) = 0 \implies (2x-1)(x+2) = 0 \tag{15}$$

This gives us two solutions: $x = 1/2, -2$. However, the domain of $y = log_2 x$ does not extend to negative x so the only valid solution is $x = 1/2$.

The answer is B!

2.2.5 Qu5: Gravitation

The general equation of the gravitation field strength g is:

$$g = \frac{GM}{r^2} \tag{16}$$

The ratio of the Earth's surface field strength to the field strength a distance R away from the centre is then given by:

$$\frac{g_E}{g_R} = \frac{10}{2} = (\frac{R}{R_E})^2 \implies \frac{R}{R_E} = \sqrt{5} \tag{17}$$

Therefore, $R_E = R/\sqrt{5}$ and the answer is D!

2.2.6 Qu6: Periodic Functions

For this question we have to think carefully about what the boundary cases are.

As $x \longrightarrow \infty \implies \frac{100}{x} \longrightarrow 0 \implies y \longrightarrow 0$. This is what happens right at the upper boundary of x.

Conversely $x = 0.1 \implies \frac{100}{x} = 1000$. We know that $y = sin\omega$ is maximum for $\omega = 90 + 360n$, for integer values of n. That is, after the first maxima occurs at 90°, maxima will come by periodically at 360 intervals. We just need to count up how many occur before we hit the 1000° mark.

$$90 + 360n = 1000 \tag{18}$$

$$n = 2.5 \tag{19}$$

So 2.5 maxima occur after the first at 90. That is 3 maxima occur in the specified range.

The answer is C!

2.2.7 Qu7: EM Waves

The key is to understand the type of radiation emitted in these cases, then the rest is simple!

1. Visible light
2. Microwave
3. Gamma
4. Infra-red
5. Radio

So from shortest to longest:

$$3, 1, 4, 2, 5 \tag{20}$$

The answer is A.

2.2.8 Qu8: Probabilities and Combinations

Perhaps the easiest way to do this question is to list the different combinations. So here they are:

1. YY YY
2. ZZ ZZ
3. YY ZZ
4. ZZ YY

We want the probability that there is a pair of Z given that we know there is also a pair of Y in the box. The probability of any given outcome (YY or ZZ) is $\frac{1}{2}$.

$$P(Z|Y) = \frac{P(Z \cap Y)}{P(Y)} = \frac{(1/2 \times 1/2) \times 2}{\frac{3}{4}} = \frac{2}{3} \tag{21}$$

You could have also reached this by observation of the 4 cases listed above; three contain YY and of the three, two contain ZZ hence 2/3. Either way, answer is D!

2.2.9 Qu9: Groups

This one is just a case of arranging the students in groups of 6 until their practicals are complete. Note some students have to do multiple practicals and they can't complete multiple practicals in the same session.

Let's label three groups based on how many practicals the students must complete.

- Group A: 2 students
  Sort once
- Group B: 4 students
  Sort twice
- Group C: 4 students
  Sort three times

The subscripts will keep tabs on what group the students are coming from. The most economical arrangement will be:

1. $2_A + 4_B$
2. $4_B + 2_C$
3. $4_C + 2_C$
4. 4_C

So 4 lessons are needed - B!

2.2.10 Qu10: Sequences

Just a sequence of prime numbers. The next one on the list should be 61. The answer is A!

2.2.11 Qu11: Scale Factors

Pay particular close attention to diameter scales as opposed to volume scale in this question! In terms of diameters, we need to estimate the diameter of an atom first.

$$d_{atom} \longrightarrow 10^{-10} \tag{22}$$

$$d_{stone} \longrightarrow 10^{-1} \tag{23}$$

When the stone is hit, it splits into 3 pieces of equal *volume*:

$$\frac{V_{new}}{V_{old}} = \frac{1}{3} \implies \frac{d_{new}}{d_{old}} = \frac{1}{3^{1/3}} \tag{24}$$

Eventually after bashing the stone n times, we should end up with a $d \sim d_{atom}$. Let's write this as an equation!

$$d_{stone,0} \frac{1}{(3^{1/3})^n} = d_{atom} \implies \frac{1}{10}\frac{1}{(3^{1/3})^n} = \frac{1}{10^{10}} \implies 3^{n/3} = 10^9 \tag{25}$$

$$\frac{n}{3} ln3 = 9 ln10 \tag{26}$$

$$n = \frac{27 ln10}{ln3} \approx 56 \tag{27}$$

The answer is D!

2.2.12 Qu12: Transformations

It is a good idea to sketch each individual part of this transformation (see Figure 2.

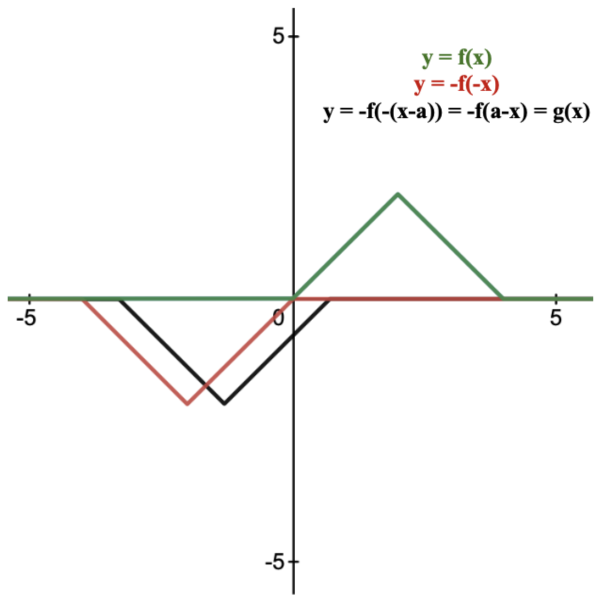

Figure 2: Qu12

The following is a discussion summarising the transformations:

- $f(x) \longrightarrow -f(x)$ is a *reflection along the x axis*.

  This actually looks like A.

- $-f(x) \longrightarrow -f(-x)$ a *reflection along the y axis*.

- $-f(-x) \longrightarrow g(x) = -f(a-x)$ is a translation or *shift by a to the right* (as $x \longrightarrow (x-a)$) given by vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

  The graph is nudged to the right by $a < b$ so part of the base actually doesn't fully make it past the y axis.

  This looks like the situation in E.

The answer is E!

3 Section B: Long Answer Questions

3.0.1 Qu13: Simultaneous Equations

We'll start by translating the information we have been given into three expressions which are equal to each other.

1. $2m_a + 3m_b$
2. $5m_a + m_c$
3. $2m_a + m_b + m_c$

Now we will try to take combinations of pairs of equations to get $m_{b,c} = f(m_a)$.

Let's start by setting $(1) = (3)$.

$$2m_a + 3m_b = 2m_a + m_b + m_c \implies m_c = 2m_b \tag{28}$$

That's handy! Next let's do $(1) = (2)$.

$$2m_a + 3m_b = 5m_a + m_c \implies 3m_b = 3m_a + m_c \tag{29}$$

Substitute $m_c = 2m_b$ from before and we arrive at our final answer!

$$m_b = 3m_a \tag{30}$$

$$m_c = 6m_a \tag{31}$$

3.0.2 Qu14: Trigonometry

Start by applying the identity,

$$cos^2(\theta) = 1 - sin^2(\theta) \tag{32}$$

, then just follow through with the algebra.

$$4(1 - sin^2(\theta)) + 2(\sqrt{3} - 1)sin(\theta) = 4 - \sqrt{3} \tag{33}$$

$$4sin^2(\theta)) - 2(\sqrt{3} - 1)sin(\theta) - \sqrt{3} = 0 \tag{34}$$

$$2sin(\theta)(2sin(\theta) - \sqrt{3}) + 1(2sin(\theta) - \sqrt{3}) = 0 \tag{35}$$

$$(2sin(\theta) - \sqrt{3})(2sin(\theta) + 1) = 0 \tag{36}$$

Solutions are:

$$sin(\theta) = \frac{\sqrt{3}}{2}, \frac{-1}{2} \tag{37}$$

In the $0 \leq \theta \leq 360°$ range, $sin(\theta) = \frac{\sqrt{3}}{2} \implies \theta = 60°, 120°$ and $sin(\theta) = \frac{-1}{2} \implies \theta = 210°, 330°$. The final solutions are therefore:

$$\theta = 60°, 120°, 210°, 330° \tag{38}$$

3.0.3 Qu15: Springs, Energy and Kinematics

Note that all of the heights h_i are measured in relation to the bottom of the mass-spring system. This is the base of the spring section.

(a) Finding the maximum height of the bounce

Two ways to do this: the energy method or SUVAT method. The former is probably simpler/ quicker. We'll outline both below.

Method 1: The Energy Method

Total energy of the system at any given point is $U_{tot} = GPE + KE + EPE$ (sum of gravitational potential energy, kinetic energy and elastic potential energy). Note that the GPE is measured based on the position of the ball only as the spring is assumed to be massless. Initially:

$$U_0 = mg(h_0 + L) + \frac{1}{2}k(\Delta L)^2 \tag{39}$$

At the maximum height, KE is zero and the spring is back to its uncompressed length so EPE must also be 0. This means the energy at h_{max} is:

$$U_{fin} = mg(h_{max} + L_0) \tag{40}$$

Applying *Conservation of Energy* ($U_0 = U_{fin}$):

$$mg(h_{max} + L_0) = mg(h_0 + L) + \frac{1}{2}k(\Delta L)^2 \tag{41}$$

$$h_{max} = h_0 + (L - L_0) + \frac{1}{2}\frac{k}{mg}(\Delta L)^2 \tag{42}$$

As $L_0 - L = \Delta L$,

$$h_{max} = h_0 - \Delta L + \frac{k}{2mg}(\Delta L)^2 \tag{43}$$

Method 2: SUVAT

Write in what you know for going down and going up (to the maximum height) scenarios. We need to find an expression for v_{up} which is the initial velocity

Going Down	Going Up
$s = h_0 + L$	$h_{max} + L_0$
$u = 0$	$u = v_{up}$
$v = v_d$	$v = 0$
$a = g$	$a = -g$
$t =$	$t =$

from the point the spring decompresses at the ground to throw the mass-spring system up towards the maximum height.

We can use energy conservation (just before compression vs just after compression). The ball's KE at the ground (going down) plus the elastic potential energy of the spring is all converted to the KE of the ball after the decompression (to move it up).

$$\frac{1}{2}mv_d^2 + \frac{1}{2}k(\Delta L)^2 = \frac{1}{2}mv_{up}^2 \tag{44}$$

$$v_{up}^2 = v_d^2 + \frac{k}{m}(\Delta L)^2 \tag{45}$$

We need expressions for v_{up} and v_d. Let's use $v^2 = u^2 + 2as$ (SUVAT equation) to the going up and down cases (see table) by turn.

$$v_d^2 = 2g(h_0 + L) \tag{46}$$

$$v_{up}^2 = 2g(h_{max} + L_0) \tag{47}$$

Applying these both back into Equation 45:

$$(h_0 + L) + \frac{k}{2mg}(\Delta L)^2 = (h_{max} + L_0) \tag{48}$$

Rearrange for h_{max} and apply $\Delta L = (L_0 - L)$.

$$h_{max} = h_0 - \Delta L + \frac{k}{2mg}(\Delta L)^2 \tag{49}$$

Hooray! We got the same answer!

(b) At a horizontal distance x_0 away from the launch point, we want to hit the maximum vertical height h_W (i.e. vertical velocity $v = 0$). We can write down the following:

$$x = v_0 t \tag{50}$$

Where $t = t_{down} + tup$ as the system must drop first and then bounce up. We can use the SUVAT equations $s = ut + \frac{1}{2}at^2$ and $s = vt - \frac{1}{2}at^2$ in the vertical direction to find expressions for t_{down} (initial velocity is zero) and tup (final velocity is zero) respectively.

$$t_{down} = (\frac{2h_0}{g})^{1/2} \tag{51}$$

$$t_{up} = (\frac{2h_{max}}{g})^{1/2} \tag{52}$$

Thus:

$$x = \frac{2v_0}{g}[\sqrt{h_0} + \sqrt{h_{max}}] \tag{53}$$

Where our answer from (a) defines h_{max}.

3.0.4 Qu16: Differential Equations

We have two differential equations and essentially have to just integrate two polynomials. Let's take it step by step.

Step 1

$$\frac{df(x)}{dx} = -2x - x^{\frac{1}{2}} + \frac{1}{3} \tag{54}$$

$$f(x) = \int -2x - x^{\frac{1}{2}} + \frac{1}{3}\,dx \tag{55}$$

$$f(x) = -x^2 - \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}x + c \tag{56}$$

Apply the boundary condition $f(1) = -1$:

$$f(1) = -1 - \frac{2}{3} + \frac{1}{3} + c = -1 \implies c = \frac{1}{3} \tag{57}$$

This gives us a final expression for $f(x)$.

$$f(x) = -x^2 - \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}x + \frac{1}{3} \tag{58}$$

Step 2

Now we tackle the second differential equation the same way:

$$\frac{dg(x)}{dx} = -x^2 - \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}x + \frac{1}{3} \tag{59}$$

$$g(x) = \int -x^2 - \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}x + \frac{1}{3}\,dx \tag{60}$$

$$g(x) = -\frac{x^3}{3} - \frac{4}{15}x^{\frac{5}{2}} + \frac{1}{6}x^2 + \frac{1}{3}x + c \tag{61}$$

To find c again apply the given boundary condition $g(1) = 0$:

$$g(1) = -\frac{1}{3} - \frac{4}{15} + \frac{1}{6} + \frac{!}{3} + c = 0 \implies c = \frac{1}{10} \tag{62}$$

So finally we have $g(x)$!

$$g(x) = -\frac{x^3}{3} - \frac{4}{15}x^{\frac{5}{2}} + \frac{1}{6}x^2 + \frac{1}{3}x + \frac{1}{10} \tag{63}$$

3.0.5 Qu17: Resolving Forces, Circular Motion

(a) The angle $\alpha_{min} > 0$ comes about due to the fact that the bike and the rider cannot just be floating completely vertically right next to the wall. The wheels must make contact with the wall at some point and if we were to push the bike as flat as possible against the wall, the bit of the handle bars that stick out will result in this non-zero minimum angle (Figure 3).

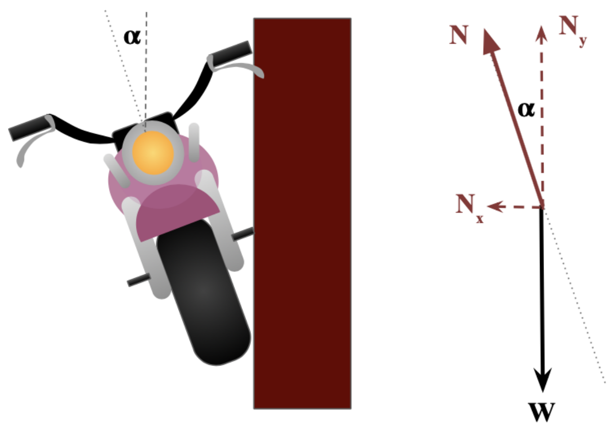

Figure 3: Qu17

To maintain a horizontal trajectory, the bike must be acted on by a force which balances the force of gravity on it (so it doesn't fall) but also facilitates circular motion. This is also shown in the figure (right).

(b) Explicit definitions of the balancing forces are needed.

$$N_y = W \implies mg = Ncos(\alpha) \tag{64}$$

The centripetal force keeps the bike at a constant radius R:

$$N_x = Nsin(\alpha) = \frac{mv^2}{R} \tag{65}$$

Dividing Equation 65 by Equation 64 allows us to find v_{min}.

$$tan(\alpha) = \frac{v^2}{gR} \tag{66}$$

$$v_{min} = \sqrt{gR\,tan(\alpha_{min})} \tag{67}$$

(c) The most tedious part: substituting the given numbers into the previous equation gives

$$v_{min} = 4.81\ ms^{-1}\ (3\ s.f.) \tag{68}$$

3.0.6 Qu18: Inequalities

Beware! Dividing the denominator over and calling it a day is a rookie mistake; you will lose solutions! Instead we want to multiply each side by the square of the denominator and then factorise and solve:

$$\frac{2x^2+3x-2}{2x^2-3x-2} > 0 \tag{69}$$

$$(2x^2+3x-2)(2x^2-3x-2) > 0 \tag{70}$$

$$[(2x-1)(x+2)][(2x+1)(x-2)] > 0 \tag{71}$$

So the solutions for the "equal" case are:

$$x = \pm\frac{1}{2}, \pm 2 \tag{72}$$

We want the positive regions of the curve in Equation 70. These regions are super easy to identify with the aid of a quick sketch if in doubt (Figure 4).

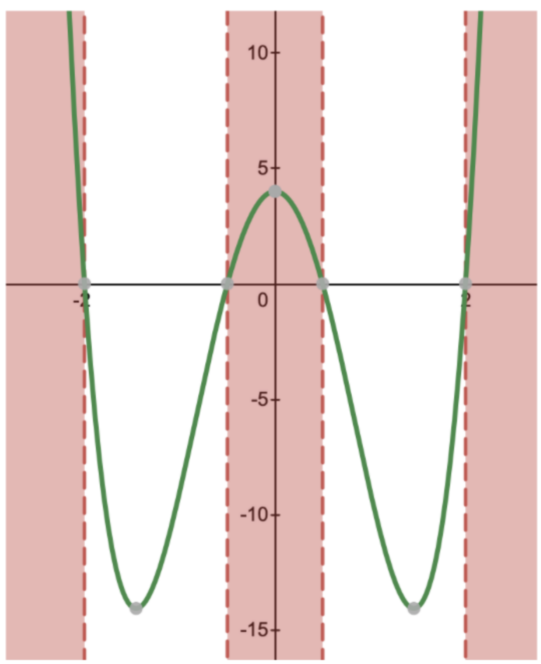

Figure 4: Qu18: Quartic

Finally, this means the overall regions that fit our inequality are:

$$x < -2, -\frac{1}{2} < x < \frac{1}{2}, x > 2 \tag{73}$$

Extra Note

In this case, sketching the quartic was by the simplest method to identify the regions you want. However, if this is not your cup of tea you can take the *sketching quadratics* route. We want the left side of Equation 71 to be positive, so that means, if we considered the numerator (A) and denominator (B) as two curves, we need to identify regions where either:

- Both curves are positive

  i.e. for (A) $x < -2, x > \frac{1}{2}$
  for (B) $x > 2, x < -\frac{1}{2}$

- Both curves are negative

  i.e. for (A) $-2 < x < \frac{1}{2}$
  for (B) $-\frac{1}{2} < x < 2$

Using a sketch similar to that in Figure 5 should land you at the same answer as before!

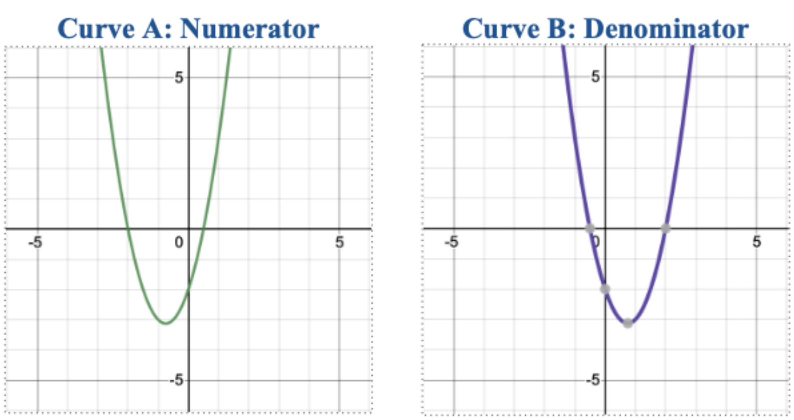

Figure 5: Qu18: Quadratics

You can also just think about positive and negative combinations of the individual elements in Equation 71; there are many ways to identify the regions, they should all give you the same answer!

3.0.7 Qu19: Dimensional Analysis

Define how you want to label your dimensions to keep everything organised:

- Mass: $[M]$
- Length: $[L]$
- Time: $[T]$

Let's consider the three quantities we have and figure out how to express them in terms of the above.

- Pressure $P = \frac{F}{A} = \frac{kgms^{-2}}{m^2} = kgm^{-1}s^{-2}$

$$[M][L]^{-1}[T]^{-2} \tag{74}$$

- Density $\rho = \frac{m}{V} = kgm^{-3}$

$$[M][L]^{-3} \tag{75}$$

- Velocity $v = \frac{s}{t} = ms^{-1}$

$$[L][T]^{-1} \tag{76}$$

Let's put it altogether to find an expression for the dimensions of E_u.

$$[E_u] = ([M][L]^{-1}[T]^{-2})^a \; ([M][L]^{-3})^b \; ([L][T]^{-1})^c \tag{77}$$

Consider the powers of each of $[M]$, $[L]$ and $[T]$ making up $[E_u]$ and setting them equal to 0 because we know E_u is dimensionless:

1. $[M]$: $a + b = 0$

$$a = -b \tag{78}$$

2. $[L]$: $-a - 3b + c = 0 \implies -a + 3a + c = 0$

$$c = -2a \tag{79}$$

3. $[T]$: $-2a - c = 0$

$$c = -2a \tag{80}$$

Combining to find ratio:

$$a : b : c = -1 : 1 : 2. \tag{81}$$

3.0.8 Qu20: Polynomials Graphs and Tangents

We need to first find what values of m are possible and then figure out the coordinates of the intersections accordingly. Our plan of action is thus:

1. Find allowed values of m

   A tangent will only touch the curve it is tangential to. The quadratic we have to solve should only have one solution i.e. $b^2 - 4ac = 0$. We exploit this fact to give us possible values of m.

2. Find coordinates of intersections

   The gradient at the point of intersection must be the same for both curve and line. We will differentiate and equate gradients. This should give us x and then we can easily find y.

Task 1: Find m?

Equate both equations, use $b^2 - 4ac = 0$, solve for m.

$$9x^2 + 6x - 7 = m(3x - 2) \tag{82}$$

$$9x^2 + (6 - 3m)x - (7 - 2m) = 0 \implies (6 - 3m)^2 + 36(7 - 2m) = 0 \tag{83}$$

$$36 - 36m + 9m^2 + 36(7 - 2m) = 0 \tag{84}$$

$$\frac{1}{4}m^2 - 3m + 8 = 0 \implies m^2 - 12m + 32 = 0 \tag{85}$$

$$(m - 8)(m - 4) = 0 \tag{86}$$

Allowed values of m:

$$m = 4, 8 \tag{87}$$

Task 2: Coordinates of Intersection

You can either differentiate the two curves individually and equate gradients or you can just solve for the stationary point of the combined graph in Equation 83. We will do the latter here:

$$f(x) = 9x^2 + (6 - 3m)x - (7 - 2m) \implies \frac{df(x)}{dx} = 18x + (6 - 3m) \tag{88}$$

$$\frac{df(x)}{dx} = 18x + (6 - 3m) = 0 \implies x = \frac{3m - 6}{18} = \frac{m - 2}{6} \tag{89}$$

Using the allowed m in Equation 87 and substituting into one of the curve equations to find corresponding y, we get our intersection points:

$$(\frac{1}{3}, -4), (1, 8). \tag{90}$$

3.0.9 Qu21: Reference Frames and Orbital Motion

(a) Relative Motion of the Moons

The longer the period, the slower the motion. Therefore, the angular frequencies of the three rotating bodies (astronomer on the equator of Mars, Phobos and Deimos) are $\omega_M, \omega_P = 3\omega_M, \omega_D = 4/5\omega_M$ respectively. That is:

$$\omega_D < \omega_M < \omega_P \tag{91}$$

This is a crude visualisation but imagine three disks rotating at the same frequency as that of Mars, Phobos and Deimos. Now we want to consider the situation in Mars' reference frame, so we will wind the disk back to ”cancel out” the rotation of Mars making it stationary. This means that at a given point of observation, Deimos would be observed to move in the opposite direction to the way we were viewing the whole system, while Phobos continues moving in the same direction as before. This means **according to the astronomer, Deimos and Phobos should be moving in opposite directions**. A ”linear” version of this argument is shown in Figure 6.

Figure 6: Qu21

(b) Phase Variation

The observation of the phases is based on the ability of an observer on the surface of Mars to view the light reflected by the Moons due to the Sun shining on them. The relative positions of the observer, Moon and Sun therefore directly relate to the observed phase.

Assuming that the relative change in position of the Mars-Moon system around the Sun is small compared to the motion of the Moons in relation to Mars, we mainly need to consider how the orbital frequency could impact our observations of phase.

- New Moon
  When the moon is closest to the Sun, all of the light from the Sun hits the face that is hidden from the astronomer on Mars: the dark side of the moon. He/she thus observes a new moon as there is no light to be reflected in this direction.

- Full Moon
  The converse of the previous is when the moon is furthest away from the Sun and all of the light from Sun hits the face of the Moon that faces the astronomer on Mars. The light is reflected from the surface of the Moon and is directed right to the observer and so the entire lit up half can be seen in the sky.

Between new moon to full moon, the moon waxes (from crescent to quarter to gibbous) and the from full moon to new moon it wanes (from gibbous to quarter to crescent).

The moons will be observed to get through all of their phases within their respective orbital periods. This means Phobos will quickly cycle through its phases while Deimos will appear to go through its phases much more slowly, taking 5/4 Martian days.

(c) Seeing Phobos rise and set in the same night?

This would require atleast one orbital period to be complete while the astronomer still experiences night on Mars (otherwise you won't be able to actually see where the moon is). Definitely possible because the orbital period is so short (less than half a Martian day)!

3.0.10 Qu22: Mathematical Problem Solving

Let us express the numbers we have been given in terms of N then equate the two sides.

- 1101
$$1N^0 + 0N^1 + 1N^2 + 1N^3 \implies N^3 + N^2 + 1 \tag{92}$$
- 313
$$3N^0 + 1N^1 + 3N^2 \implies 3N^2 + N + 3 \tag{93}$$
- 344
$$4N^0 + 4N^1 + 3N^2 \implies 3N^2 + 4N + 4 \tag{94}$$

$$N^3 + N^2 + 1 - (3N^2 + N + 3) = 3N^2 + 4N + 4 \tag{95}$$
$$N^3 - 2N^2 - N - 2 = 3N^2 + 4N + 4 \tag{96}$$
$$N^3 - 5N^2 - 5N - 6 = 0 \tag{97}$$
$$(N-6)(N^2 + N + 6) = 0 \tag{98}$$

This only has one solution as the quadratic has no real roots (check $b^2 - 4ac$). The base is:
$$N = 6. \tag{99}$$

3.0.11 Qu23: Refraction and Reflection

This question will use:

- Law of reflection

$$\theta_{inc} = \theta_{refl} \tag{100}$$

- Snell's Law

$$n_1 sin(\theta_1) = n_2 sin(\theta_2) \tag{101}$$

The incoming light ray has the following trajectory:

Refraction at liquid boundary $\longrightarrow$ Reflection at mirror $\longrightarrow$ Refraction/ Reflection at liquid boundary $\longrightarrow$ etc...

For the light ray to not leave the tank after the first reflection, total internal reflection must occur when the liquid-air boundary approaches after the first reflection. A sketch of the "critical" situation is provided in Figure 7.

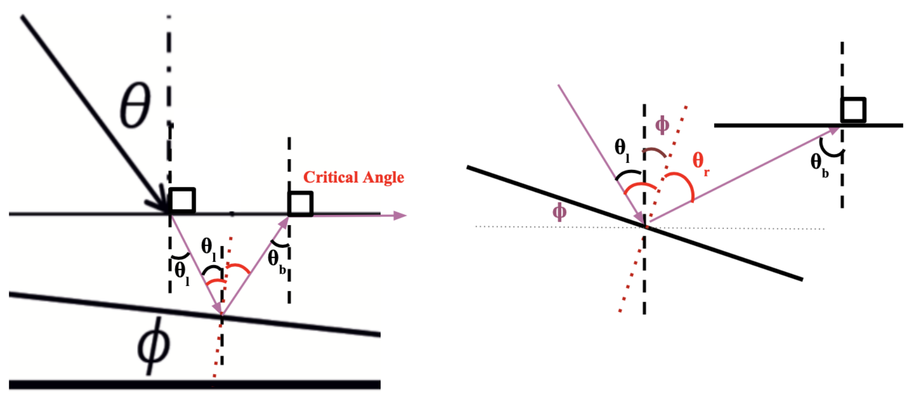

Figure 7: Qu23

The endgame is an expression for θ in terms of the refractive indices we have and the angle ϕ. We will start by writing down what we know.

Snell's Law at the first liquid-air boundary:

$$n_a sin(\theta) = n_l sin(\theta_l) \implies sin(\theta) = \frac{n_l}{n_a} sin(\theta_l) \tag{102}$$

The following relations can be inferred from the "zoomed" diagram in Figure **??**.

$$\theta_r = \theta_l + \phi \tag{103}$$

$$\theta_b = \theta_r + \phi = \theta_l + 2\phi \tag{104}$$

Now the critical condition at the boundary after the reflection via Snell's Law gives us:

$$n_l sin(\theta_b) = n_a \implies n_l sin(\theta_l + 2\phi) = n_a \quad (105)$$

This gives us:

$$\theta_l = 28.6° \ (3\ s.f.) \quad (106)$$

Applying in Equation 102:

$$\theta = 39.6° \ (3\ s.f.) \quad (107)$$

3.0.12 Qu24: Geometry

Basics:

- Area of a Sector

$$A_{sec} = \frac{1}{2} r^2 \theta \quad (108)$$

- Area of a Triangle

$$A_{\Delta} = \frac{1}{2} ab sin(C) \quad (109)$$

We will find the areas of the two sectors and then take away the area of the triangles. First drawing a quick diagram to identify angles (Figure 8).

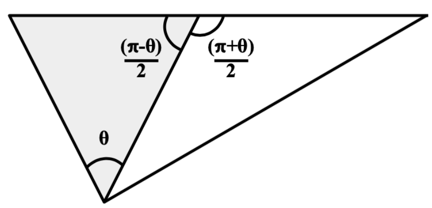

Figure 8: Qu24

From Equation 108, the total area of the two sectors:

$$A^{Tot}_{sec} = \frac{\theta}{2} x^2 + (\frac{\pi + \theta}{4}) x^2 \quad (110)$$

Area of the triangles from Equation 109:

$$A^{Tot}_{\Delta} = \frac{1}{2} x^2\ sin(\theta) + \frac{1}{2} x^2\ sin(\frac{\theta + \pi}{2}) = \frac{1}{2} x^2\ sin(\theta) + \frac{1}{2} x^2\ cos(\frac{\theta}{2}) \quad (111)$$

The shaded area $A_{shaded} = A^{Tot}_{sec} - A^{Tot}_{\Delta}$.

$$A_{shaded} = \frac{x^2}{4}[(\pi + 3\theta) - 2(sin(\theta) + cos(\frac{\theta}{2}))] \quad (112)$$

3.0.13 Qu25: Millikan Oil Drop Experiment

(a) We'll define some terms first quickly to make things extra clear:

- Force due to Electric Field (E): F_Q

$$F_Q = QE \tag{113}$$

- Weight: F_W

$$F_W = \frac{4}{3}\pi r^3 g \rho_{oil} \tag{114}$$

- Drag Force: F_D

$$F_D = 6\pi\eta r v_t \tag{115}$$

- Buoyant Force: F_B

$$F_B = \frac{4}{3}\pi r^3 g \rho_{air} \tag{116}$$

The switch is open so we don't care about the electric field term. At terminal velocity, the forces are all balanced (Newton's 1st Law):

$$F_W - F_D - F_B = 0 \tag{117}$$

$$\frac{4}{3}\pi r^3 g(\rho_{oil} - \rho_{air}) - 6\pi\eta r v_t = 0 \tag{118}$$

(b) Introduce the electric field term (Equation 113 and remove the drag term (as the drop is stationary so $F_D = 0$).

$$F_W - F_B - F_E = 0 \tag{119}$$

$$\frac{4}{3}\pi r^3 g(\rho_{oil} - \rho_{air}) - QE = 0 \tag{120}$$

(c) Equations 118 and 120 can be written as follows:

$$\frac{4}{3}\pi r^3 g(\rho_{oil} - \rho_{air}) = 6\pi\eta r v_t \tag{121}$$

$$\frac{4}{3}\pi r^3 g(\rho_{oil} - \rho_{air}) = QE \tag{122}$$

Dividing these two equations gives us an expression for r.

$$QE = 6\pi\eta r v_t \implies r = \frac{QE}{6\pi\eta v_t} \tag{123}$$

Substitute this into Equation 120.

$$QE = \frac{4}{3}\pi g(\rho_{oil} - \rho_{air})(\frac{QE}{6\pi\eta v_t})^3 \tag{124}$$

Rearrange for Q:

$$Q^2 = \frac{3}{4}\frac{\pi^2(6\eta v_t)^3}{E^2 g(\rho_{oil} - \rho_{air})} \tag{125}$$

$$Q = \frac{9\sqrt{2}\pi}{E}[\frac{(\eta v_t)^3}{g(\rho_{oil} - \rho_{air})}]^{1/2} \tag{126}$$

Note this answer may be expressed in different ways so might end up looking different depending on how you simplified your answer.

3.0.14 Qu26: Polynomial Graphs

(a) Polynomial Graph Sketching

This is a pretty simple start.

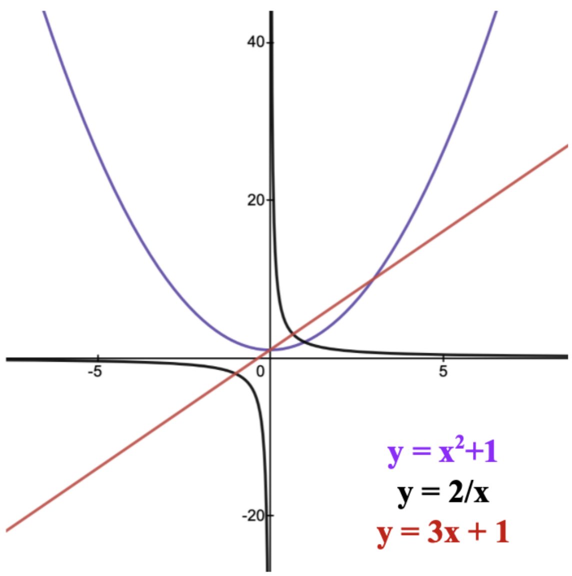

Figure 9: Qu26 a

(b) There are three graphs and we must compare each to the others.

- Linear and Quadratic

$$x^2 + 1 = 3x + 1 \implies x^2 - 3x = 0$$

The x intercepts are:

$$x = 0, 3. \tag{127}$$

- Linear and Reciprocal

$$3x + 1 = \frac{2}{x} \implies 3x^2 + x - 2 = 0$$

$$(3x - 2)(x + 1) = 0$$

The x intercepts are:

$$x = \frac{2}{3}, -1 \tag{128}$$

- Quadratic and Reciprocal

$$\frac{2}{x} = x^2 + 1 \implies x^3 + x - 2 = 0$$

Look for $f(x) = 0$ like $f(1) = 0$.

$$(x + 1)(x^2 + x - 2 = 0$$

Can see only one solution from plot too so intercept:

$$x = 1. \tag{129}$$

Remember to go back to the plot and label the x values of the intersection points (we haven't done this here)!

(c) Identify the region specified first so that we can figure out what we are integrating and the limits of the integrals.

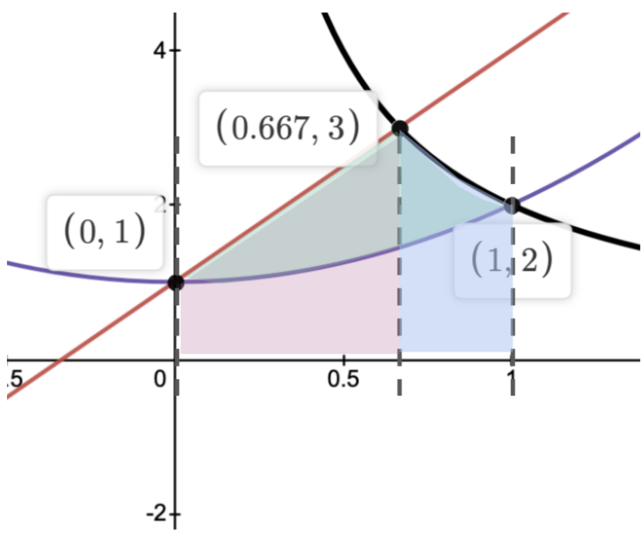

Figure 10: Qu26 c

We want the green area. To get to it we need to find two different areas and combine.

1. The pink area's green contribution

$$\int_0^{\frac{2}{3}} (3x+1) - (x^2+1)\, dx \implies [\frac{3}{2}x^2 - \frac{x^3}{3}]_0^{\frac{2}{3}} \tag{130}$$

2. The blue area's green contribution

$$\int_{\frac{2}{3}}^1 2x^{-1} - (x^2+1)\, dx \implies [2ln(x) - \frac{x^3}{3} - x]_{\frac{2}{3}}^1 \tag{131}$$

Sum the two:

$$[\frac{3}{2}x^2 - \frac{x^3}{3}]|_{x=\frac{2}{3}} - \frac{1}{3} - 1 + \frac{x^3}{3}|_{x=\frac{2}{3}} + \frac{2}{3} - 2ln(\frac{2}{3}) \tag{132}$$

$$Area = 2ln(\frac{3}{2}) \tag{133}$$

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