Unofficial Solutions to Interview Questions: 2020

University of Oxford Admissions Interview

Physics

PhysOx Initiative
2020

Foreword 0

The questions and solutions provided in this manual are intended as a resource to aid preparation for the Oxford physics interviews. We have tried to create problems which ultimately use nothing more advanced than A-level knowledge but could facilitate interesting discussions in an interview setting. We are not claiming that these questions may come up in an interview and neither are we claiming that the difficulty of these questions are representative of that of all real interview questions. We merely wanted to invite you to consider the applications of the physics and mathematics you may have come across in your course and assimilate them together to unravel the dynamics of some very physical scenarios. We also note that in a real interview the problems are not meant to be solved in one go. There will be an active to-and-forth between you and the (typically) two other tutors who will be interviewing you, rather than a setup where you are expected to quietly work through your answer, without any input, on a board.

Finally, for all those who have been invited for an interview, we wish you the best of luck!

Question 1

(a) Consider a ball of mass M placed on a track that begins at height h. Find the minimum value of h such that the ball can make it around a loop of radius R as shown in figure 1.

(b) What assumptions did you make during your calculations in part a?

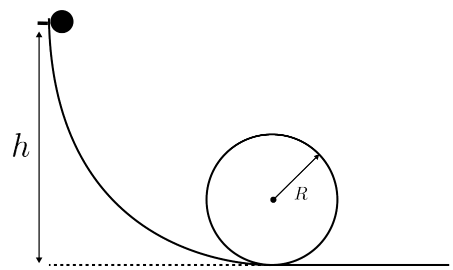

Figure 1: Loop-de-Loop

Solution 1

(a) First we find the minimum speed required to make it around the loop without losing contact. To do this we first consider the forces acting on the ball as it travels around the loop.

$$\sum F = \vec{N} + m\vec{g},$$

where $\vec{N}$ is the normal force. If we want the ball to remain in contact with the track we shall constrain the motion to that of a circle. Hence the resultant force must be equal to the centripetal force. In particular, we want to consider the speed required to stay in contact with the loop when the ball is at its apex (top of the loop).

$$\frac{mv^2}{r} = N + mg$$

Since the gravitational force is fixed our speed around loop is determined by the normal force. The lowest speed the ball can go around the loop is when the ball is just in contact with the loop and $N = 0$.

$$v_{min} = \sqrt{Rg}$$

The gravitational potential energy of the ball on the hill must be equal to the kinetic energy plus the potential energy of the ball at the apex of the loop.

$$mgh = 2Rmg + \frac{1}{2}mRg$$

Solving for the height we get:

$$h = \frac{5g}{2}$$

(b) We assumed that there is no friction and that the ball does not roll.

Question 2

Calculate the effective spring constant for each system of mass-less springs shown in figure 2, where k_1 and k_2 are the spring constants of the respective springs.

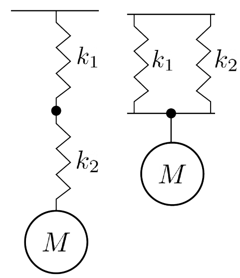

Figure 2: Spring systems

Solution 2

The force that a spring exerts on an object is given by Hooke's law:

$$F = kx,$$

where k is the spring constant. For springs in series we can derive the effective spring constant using the following two facts:

- The force on each spring is the same.
- The total displacement is given by the sum of the individual displacements. This can be written as $x_{total} = x_1 + x_2$.

Re-arranging Hooke's law for the displacement x we can write the total deformation of the spring system as:

$$x_{total} = \frac{F}{k_1} + \frac{F}{k_2} = \frac{F}{k^S_{eff}}$$

Hence the effective spring constant for the series system of springs is:

$$k^S_{eff} = \frac{k_1 k_2}{k_1 + k_2}$$

Now we consider the parallel system of springs. We assume that the mass M uniformly applies its force on the bar such that when it is connected to both springs (as shown in the figure 2) and the mass, the mass-bar system remains parallel. For this situation the force each spring exerts will be different but they will have the same displacement.

$$F = F_1 + F_2 = (k_1 + k_2)x = k^P_{eff}x$$

From this we can immediately extract the effective spring constant as:

$$k^P_{eff} = k_1 + k_2$$

Question 3

Evaluate the following derivative:

$$\frac{d}{dx}x^x$$

Solution 3

It helps to start by writing this in a more familiar form:

$$y = x^x$$

Now we can use a bit of implicit differentiation. For those who are unfamiliar with this, it is just an extension of chain rule. We will do it taking the most general approach. Note that when we have to differentiate something with the variable we want to differentiate with respect to in the power, we can take logs to "bring it down".

$$ln(y) = xln(x)$$

Implicitly differentiating the left hand side would mean doing $\frac{d(ln(y))}{dy}\frac{dy}{dx}$. You can see this clearly follows from the chain rule. The right hand side just needs a bit of product rule.

$$\frac{1}{y}\frac{dy}{dx} = ln(x) + 1$$
$$\implies \frac{d}{dx}x^x = x^x(ln(x) + 1)$$

Question 4

Consider the following potential.

$$U(x) = k(1 - e^{-x^2})$$

Show that for perturbations about the equilibrium that a system subject to this potential undergoes simple harmonic motion.

Solution 4

To demonstrate that the system will undergo simple harmonic motion under small perturbations we need to show that the force under these perturbations has the form of Hooke's law i.e. $F \propto -x$. You can compute the force from a potential using the following equation:

$$F(x) = -\frac{d}{dx}U(x)$$
$$\implies F(x) = -2kxe^{-x^2}$$

Note this is just the integral form of $U = Fx$, where U is the work done, F is the force and x is the distance.

To see how this function behaves under small perturbations, we need to utilise the Taylor series. The general expansion of a function $f(x)$ about $x = x_0$ is then given by:

$$f(x) = f(x_0) + \sum_{j=0}^{\infty} f^{(j)}(x_0)(x - x_0)^j$$

Where $f^{(j)}(x_0)$ is the j^{th} derivative of the function $f(x)$ evaluated at $x = x_0$. For our force we only need to compute the first derivative and since we are perturbing about the origin $x_0 = 0$.

$$f^{(1)}(0) = -2k$$
$$\implies F(x) - 2kx + \mathcal{O}(x^2)$$

For small perturbations we can ignore higher order terms and the resulting force reduces to Hooke's law. Hence for small perturbations about the origin, a system will undergo simple harmonic motion!

Question 5

Find the solutions to the following polynomial, where $z \in \mathbb{C}$:

$$\sum_{j=0}^{7} z^j = 0$$

(Note this is only for those who have studied complex numbers).

Solution 5

To find the roots of the complex polynomial you have to first notice that the given equation is the sum of a geometric series. If we have the sum of a series:

$$S = a + ar + ar^2 + ... + ar^n$$

Then the sum can written as:

$$S = \frac{a(1 - r^{n+1})}{1 - r}$$

Using this we can re-write our complex polynomial as:

$$\frac{1 - z^8}{1 - z} = 0$$

This equation is satisfied when $z^8 = 1$. Our original problem has now reduced to one you should be familiar with: finding the n^{th} root of unity. Consider a complex number $w = z^8$ which takes the value:

$$w = re^{i\theta}$$

We also know that a complex number is invariant under the addition of integer multiples of 2π to its argument.

$$w = re^{i\theta + i2\pi p} \qquad ; p \in \mathbb{Z}$$

Hence we can find z by taking the n^{th} root of w.

$$z = r^{\frac{1}{n}} e^{i\frac{\theta}{n} + i\frac{2\pi p}{n}}$$

By considering $w = 1$ we can determine that $r = 1$ and that $\theta_1 = 0$. To find all our solutions we let p have the following range $[0, 7]$.

$$z \in (1, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{4}}, e^{i\pi}, e^{i\frac{5\pi}{4}}, e^{i\frac{3\pi}{2}}, e^{i\frac{7\pi}{4}})$$

The astute of you may have realised that we have more solutions than we need. The fundamental theorem of algebra states that an n^{th} order polynomial will have exactly n solutions (not necessarily unique or real). To find which one of our solutions does not satisfy our polynomial consider the re-written polynomial again:

$$\frac{1-z^8}{1-z} = 0$$

From this we see that the condition $z \neq 1$ must hold. Hence we need to remove this solution from our set and we get the seven solutions we are looking for.

$$z \in (e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{4}}, e^{i\pi}, e^{i\frac{5\pi}{4}}, e^{i\frac{3\pi}{2}}, e^{i\frac{7\pi}{4}})$$

Question 6

Imagine two spheres of equal masses but different densities ($\rho_1 > \rho_2$) attached to a scale in a vacuum within a uniform gravitational field such that they balance. If this system is placed in a fluid, what will happen ? (note the diagram is not to scale.)

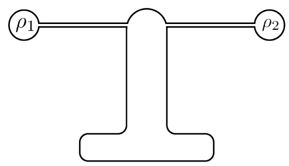

Figure 3: Scale set-up.

Solution 6

In a vacuum due to the two spheres having the same mass the scale will be balanced, but things change when we submerge this scale in a fluid.

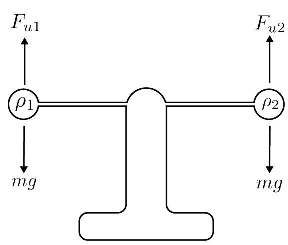

Figure 4: Scale force diagram.

We have to take into account the buoyancy force on the spheres. The buoyancy force is given by:

$$F_u = g\rho_f V_d,$$

where ρ_f and V_d are the density and volume of the displaced fluid. A scale cannot distinguish between different forces, hence due to the buoyancy force the spheres will have different effective weights. To see which effective weight will be greater we need to compare the buoyancy forces experienced by each sphere.

$$F_{u1} = g\rho_f V_{d1}$$
$$F_{u2} = g\rho_f V_{d2}$$

Since ($\rho_1 > \rho_2$) we can deduce the following:

$$\frac{\rho_1 v_{d1}}{\rho_2 v_{d2}} = 1 \implies \frac{v_{d2}}{v_{d1}} = \frac{\rho_1}{\rho_2}$$
$$\rho_1 > \rho_2 \implies v_{d2} > v_{d1}$$

Since $v_{d2} > v_{d1}$ it follows that $F_{u2} > F_{u1}$, hence if we define the effective weight as:

$$W_{effective} = mg - F_u,$$

then it follows that the scale will perceive the mass with density ρ_1 to be heavier, hence that side will tilt downwards.

Question 7

Consider the sums S and C defined below.

$$S = nsin(x) + \frac{n(n-1)}{2}sin(2x) + ... + sin(nx)$$
$$C = 1 + ncos(x) + \frac{n(n-1)}{2}cos(2x) + ... + cos(nx)$$

Show that $C + iS = (1 + e^{ix})^n$.

Solution 7

We define the following notation:

$$\binom{p}{r} = \frac{p!}{(p-r)!r!}$$

(This is your standard binomial expansion.) Using this we can re-write the sums S and C as follows:

$$S = \sum_{k=0}^{n} \binom{n}{k} sin(kx)$$
$$C = \sum_{k=0}^{n} \binom{n}{k} cos(kx)$$

Hence $C + iS$ can be written as:

$$C + iS = \sum_{k=0}^{n} \binom{n}{k} (cos(kx) + isin(kx))$$

To proceed further we can then use Euler's formula. For those who have not done complex numbers this may look a bit confusing, but for those who have, we have just decided to transition into using modulus-argument form to go from $a\cos(\theta) + ibsin(\theta)$ to $re^{i\theta}$.

$$C + iS = \sum_{k=0}^{n} \binom{n}{k} e^{ikx}$$

The binomial expansion is given by:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

If we set $a = 1$ and $b = e^{ix}$ we immediately arrive at the required equation.

$$C + iS = \sum_{k=0}^{n} \binom{n}{k} e^{ikx} = (1 + e^{ix})^n$$

Question 8

A beaker is filled with water to a height h as shown in the diagram below.

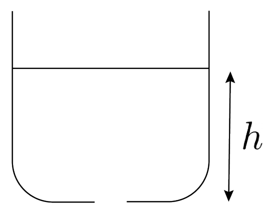

Figure 5: The Beaker.

(a) Calculate the velocity of the fluid coming out of the small opening at the bottom of the beaker of area α.

(b) Find the time taken for the beaker to empty.

Solution 8

(a) We will show you how to tackle this question in two ways. The first is via the use of questions that you may have learnt from the fluids section of your physics course and the second is via the direct application of energy conservation.

Bernoulli Method: This is the standard method of solving this problem utilising Bernoulli's equation.

$$H = P + \frac{1}{2}\rho v^2 + \rho g x = constant,$$

where P and v are the pressure and velocity of a fluid of density ρ at a height x above some reference point in a gravitational field. If we take the bottom of the beaker to be at $x = 0$ and label the surface of the water as A and the water located at the hole B, then we can write the following equations:

$$H_A = P_{atmosphere} + \rho g h$$
$$H_B = P_{atmosphere} + \frac{1}{2}\rho v^2$$

Since the Bernoulli function is constant we can equate these two equations and immediately find the velocity of the fluid leaving the hole at the bottom of the beaker as:

$$v = \sqrt{2gh}.$$

Conservation of energy: As fluid leaves through the hole, it will cause a shift in its center of mass. This shift causes a change in the gravitational potential energy (GPE) of the fluid in the beaker and this change in the energy is gained by the escaping fluid.

As a differential mass element dm of the fluid escapes through the hole at the bottom of the beaker there is small change in the height of the fluid dh. The GPE of the fluid in the beaker before and after a differential mass element of the fluid has escaped is:

$$GPE_{before} = A\rho g\frac{h^2}{2}$$
$$GPE_{after} = \frac{A\rho g}{2}(h - |dh|)^2$$
$$\Delta GPE = \rho Agh|dh| + \mathcal{O}(|dh|^2)$$

Ignoring higher order terms in dh, we set this equal to the kinetic energy of the escaping differential fluid element.

$$\frac{1}{2}(dm)v^2 = \rho Agh|dh|$$
$$dm = \alpha\rho v dt \implies v^3 = \frac{2Agh}{\alpha}|\frac{dh}{dt}|,$$

where A is the area of the fluid surface. Due to conservation of mass we can write the following:

$$\frac{dm}{dt} = -\alpha v\rho.$$

We can also write dm in terms of dh by considering differential mass element that could have been removed from the surface of the fluid to produces the same change in GPE.

$$dm = \rho Adh \implies \frac{dh}{dt} = -\frac{\alpha v}{A}$$
$$\implies v^3 = 2Aghv$$

Although this is a more involved process, we recover the solution found using the first method $v = \sqrt{2gh}$.

(b) To find the time taken, we can exploit conservation of mass (as we did in the latter part of the second solution to the previous part) to find the velocity of the escaping fluid.

$$\frac{dh}{dt} = -\frac{\alpha v}{A},$$
$$v = \sqrt{2gh}.$$

Using these equations we can set up a differential equation to solve for the time taken to empty the container.

$$\frac{dh}{dt} = -\frac{\alpha\sqrt{2g}}{A}\sqrt{h}$$
$$\implies \sqrt{h(t)} - \sqrt{h} = -t\frac{\alpha\sqrt{2g}}{2A}$$

If we define a time $t = \tau$ as the time when the container is empty, then $h(t = \tau) = 0$, therefore:

$$\tau = \frac{2A}{\alpha}\sqrt{\frac{h}{2g}}.$$

Question 9

The metabolic rate of a cell is the rate of energy expenditure per unit time by endothermic animals at rest. This process produces heat. The metabolic rate of the organism's cells are fine-tuned such that the heat produced by cells are in thermodynamic equilibrium with heat radiated by the organisms surface area, such as to maintain the body temperature of the organism at a certain value. If the metabolic rate of an organism is kept constant as we vary its size, what would happen if:

(a) An organism of constant density was rapidly shrunk?

(b) The organism of constant density was enlarged drastically?

Solution 9

If we assume our organism to be a sphere, its surface area scales as:

$$S \sim r^2$$

Since we are considering an organism of constant density, the number of cells would be proportional to the organism volume.

$$V \sim r^3 \implies \text{Number of cells} \sim r^3$$

Note that since we have said that we have a constant metabolic rate that we can't just increase our single cell heat output. The only way to change the total amount of heat generated is to change the number of cells we have.

(a) Since the volume shrinks faster than the surface area as the size of an organism is altered, as we decrease the radius of the organism it will begin to have more surface area than the number of cells it can produce heat for. This will cause the organism body temperature to decrease until it begins to freeze.

(b) For the case of enlarging an organism we have the opposite problem. As we increase its size the volume increases much faster than the surface area of the organism. This causes more heat to be produced than can be radiated, increasing the organism body temperature. In the extreme cases of a rapid and large size increase, the organism would explode.

Question 10

You are standing some distance away from a apple tree with a gun, a single apple is hanging from an extended branch about to fall. How should you aim your gun such that you hit the apple if you were to shoot at the moment that the apple starts to fall ?

Solution 10

If you aim at the apple, given that the bullet has sufficient speed to transverse the distance between you and the tree before it hits the ground the bullet will hit the apple. This is because objects fall at the same rate in a uniform gravitational field, assuming we are ignoring air resistance and other possible external forces.

Question 11

Consider the following function:

$$f(x) = \frac{4x}{x^2 + 4}$$

(a) Show that the range of this function is [-1,1].

(b) Sketch the function.

Solution 11

(a) To find the range of the given function we need to determine whether the function diverges at any point and find values of the function at any extrema (if there are any).

Since $x^2 + 4 = 0$ has no real solutions, the function does not diverge anywhere along the real axis. To find the location of the extrema we can set the first order derivative of $f(x)$ equal to zero.

$$\frac{df(x)}{dx} = 0 \implies \frac{4}{x^2 + 4} - \frac{8x^2}{(x^2 + 4)^2} = 0$$
$$\implies x = \pm 2$$

Since we have no points along the real axis where our function diverges, we can say the range of our function is determined by $[f(-2), f(2)] = [-1, 1]$.

(b) The sketch of the graph:

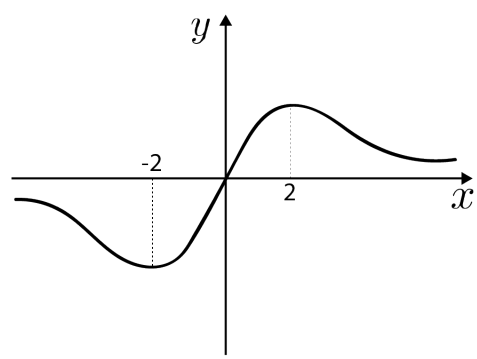

Figure 6: Function graph.

Question 12

Sketch the region defined by the equations below:

$$x^2 + y^2 - r^2 \leq 0$$
$$xy - 1 \leq 0$$

Solution 12

The region that satisfies the inequalities in our question is:

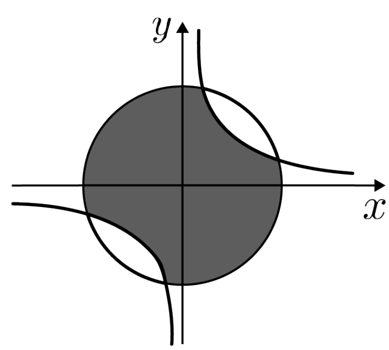

Figure 7: Inequality region.

The first inequality defines the inside of a circle of radius r. The second cuts into it as it defines the area within $y = \frac{1}{x}$.

Question 13

Consider a positive test charge placed at a point equidistant and co-linear from the centers of two fixed positive charges. If this test charge's position is perturbed from its equilibrium position along the line that connects the centers of the fixed charges, show that this test charge undergoes simple harmonic motion.

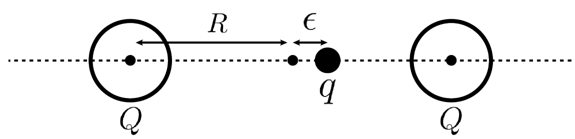

Figure 8: Charge system set-up.

Solution 13

We start by writing the resultant force on the test charge as:

$$\sum F = \frac{KQ}{(R+\epsilon)^2} - \frac{KQ}{(R-\epsilon)^2} \implies \frac{KQ}{R^2}\left(\frac{1}{(1+\frac{\epsilon}{R})^2} - \frac{1}{(1-\frac{\epsilon}{R})^2}\right)$$

To proceed further we will apply a Taylor expansion. For a function $f(x)$, the Taylor expansions about a point $x = x_0$ is given by:

$$f(x) = f(x_0) + \sum_{j=0}^{\inf} f^{(j)}(x_0)(x - x_0)^j.$$

If we set $x = \epsilon/R$ and expand about the origin $x_0 = 0$, we can get the following expansions for the functions inside the brackets of our resultant force equation.

$$(1 \pm \frac{\epsilon}{R})^{-2} = 1 \mp \frac{2\epsilon}{R} + \mathcal{O}\left(\frac{\epsilon}{R}\right)$$

Note that $\epsilon/R \ll 1$, hence we can ignore higher order terms reducing the resultant force equation to:

$$\sum F = -\frac{4KQ}{R^3}\epsilon$$

This has the form of Hooke's law and thus for small perturbations a test charge will undergo simple harmonic motion!

Question 14

Find the relationship between a,b and c for the following equation such that it has repeated roots.

$$log_b((b^x)^x) + log_a(\frac{b}{c})^x + log_a(\frac{1}{b})log_a(c) = 0 \qquad | \qquad a, b, c \in \mathbb{R}$$

Solution 14

To coax this equation into a more palpable form, we start by applying the power rule, applying it twice to the first term and once to the second term of our equation. Noting that $log_b(b) = 1$, we can write:

$$x^2 + log_a(\frac{b}{c})x + log_a(\frac{1}{b})log_a c = 0.$$

Now that our equations is in a more familiar form, consider a general second order polynomial.

$$ax^2 + bx + c = 0$$

For this polynomial to only have one solution, its discriminant must be equal to zero ($\Delta = b^2 - 4ac = 0$). Reading off the coefficients for our polynomial, we get the following equation:

$$log_a(\frac{b}{c})^2 = 4log_a(\frac{1}{b})log_a(c).$$

Applying the quotient rule:

$$(log_a(b) - log_a(c))^2 = 4log_a(b^{-1})log_a(c).$$

Applying the power rule to the right hand side and rearranging we get:

$$\begin{aligned} (log_a(b) - log_a(c))^2 &= -4log_a(b)log_a(c) \\ \implies log_a(b)^2 + log_a(c)^2 + 2log_a(b)log_a(c) &= 0 \\ \implies (log_a(b) + log_a(c))^2 &= 0. \end{aligned}$$

Finally applying the product rule to the last line we arrive at our final equation.

$$log_a(bc) = 0 \implies bc = 1 \; \forall \; a$$

Question 15

Show that:

(a) $n! = \int_0^\infty x^n e^{-x} dx$

(b) $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$

Solution 15

(a) To evaluate the integral on the right hand side of the equation consider another integral that is a function of the variable λ.

$$I(\lambda) = \int_0^\infty e^{-\lambda x} dx$$

The first and second derivatives of this function in respect to λ are:

$$\frac{dI(\lambda)}{d\lambda} = \int_0^\infty -xe^{-\lambda x} dx$$
$$\frac{d^2 I(\lambda)}{d\lambda^2} = \int_0^\infty (-1)^2 (x)^2 e^{-\lambda x} dx$$

By taking further derivatives it can be seen that:

$$\frac{d^n I(\lambda)}{d\lambda^n} = \int_0^\infty (-1)^n (x)^n e^{-\lambda x} dx$$

To proceed we need to evaluate the integral $I(\lambda)$.

$$I(\lambda) = \frac{1}{\lambda}$$

As we did with $I(\lambda)$ in its integral form, we take the first and second derivative of its evaluated form.

$$\frac{dI(\lambda)}{d\lambda} = \frac{-1}{\lambda^2}$$
$$\frac{d^2 I(\lambda)}{d\lambda^2} = \frac{(-1)(-2)}{\lambda^3}$$

By taking further derivatives it can be seen that:

$$\frac{d^n I(\lambda)}{d\lambda^n} = \frac{(-1)^n n!}{\lambda^{n+1}}$$

Equating our solutions for the derivatives of the integral and evaluated form we get:

$$\frac{(-1)^n n!}{\lambda^{n+1}} = \int_0^\infty (-1)^n (x)^n e^{-\lambda x} dx$$

Setting $\lambda = 1$ we immediately recover our sought-after solution.

$$n! = \int_0^\infty x^n e^{-x} dx$$

The technique used here is known to physicists as Feynman's trick (or differentiation under the integral sign) and to mathematicians as the Leibniz integral rule.

(b) The standard way of tackling this integral is by first squaring it.

$$I^2 = (\int_{-\infty}^\infty e^{-x^2} dx)(\int_{-\infty}^\infty e^{-x^2} dx)$$

Since x is a dummy variable we can change it to any other symbol we please without changing the value of the integral.

$$I^2 = (\int_{-\infty}^\infty e^{-x^2} dx)(\int_{-\infty}^\infty e^{-y^2} dy)$$

Which can be re-written as:

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dxdy$$

The double integral above is difficult to carry out in Cartesian co-ordinates so we change into polar co-ordinates. If this sort of question comes up in an interview, they will prompt you and tell you that this coordinate transformation needs to occur. The "area element" $dxdy$, when transformed to polar coordinates, becomes $rdrd\theta$. This information will also be given and it is not expected for you to have seen this before!

The transformed integral then looks like:

$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-(r^2)} rdrd\theta$$

$$\implies I^2 = 2\pi \int_0^\infty re^{-r^2} dr$$

We make the following substitution $s = r^2$.

$$I^2 = 2\pi \int_0^\infty \frac{1}{2} e^{-s} ds = \pi$$

Hence our original integral is evaluated as $I = \sqrt{\pi}$.

Question 16

Consider a ladder of resistors, with n levels.

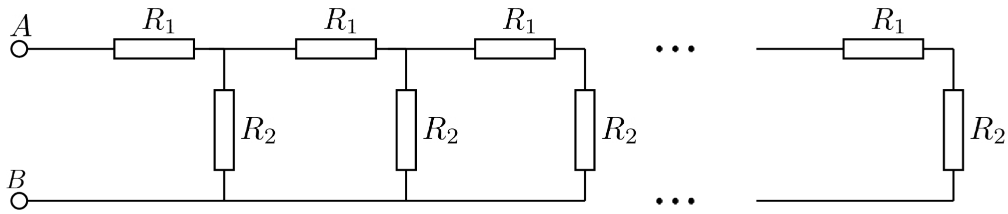

Figure 9: Resistor ladder.

(a) Show that the resistance measured between points A and B of this circuit can be found recursively using this formula:

$$R_{t(j+1)} = R_1 + \frac{R_2 R_{tj}}{R_2 + R_{tj}}$$

Where R_{tj} is the total resistance of a j level resistor ladder.

(b) Show that for $R_1 = R_2 = R$ a resistor ladder with j levels has a resistance of the form:

$$R_{tj} = R\left(1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\dots}}}\right)$$

(c) Show that the total resistance of a infinite resistor ladder for $R_1 = R_2 = R$ is given by:

$$R_{t\infty} = (\frac{1+\sqrt{5}}{2})R$$

Solution 16

(a) To tackle this problem we first consider the special cases of $n = 1$ to $n = 3$. The $n = 1$ case is simple, we just have two resistors in series, hence the total resistance of the one level resistor ladder is:

$$R_{t1} = R1 + R2$$

We can tackle the $n = 2$ case by noticing that the $n = 1$ network is embedded within it. Hence the $n = 2$ resistor network can be reduced to

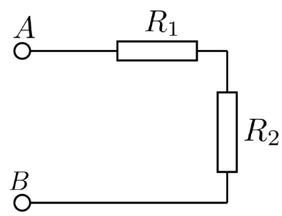

Figure 10: n=1 resistor ladder.

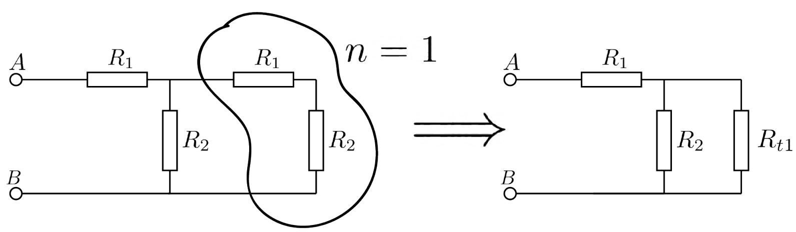

Figure 11: n=2 resistor ladder.

a simple series and parallel resistor problem. The total resistance of the parallel is given by:

$$R_{parallel} = \frac{R2R_{t1}}{R2 + R_{t1}}$$

Hence the total resistance of our 2-level resistor ladder is given by:

$$R_{t2} = R1 + \frac{R2R_{t1}}{R2 + R_{t1}}$$

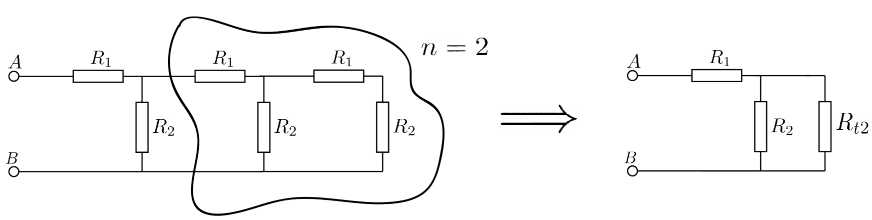

Figure 12: n=3 resistor ladder.

For the $n = 3$ case we proceed in a similar fashion to the $n = 2$ case. It can be seen that the 2-level resistor ladder is embedded within our 3-level resistor ladder network, hence we can again reduce our problem to a simple series and parallel resistor network problem. Applying the same procedure for $n = 3$ as for $n = 2$ we can write the resistance of a 3-level resistor ladder network as:

$$R_{t3} = R1 + \frac{R2R_{t2}}{R2 + R_{t2}}$$

We start to see a pattern that we can generalise to:

$$R_{t(n+1)} = R_1 + \frac{R_2R_{tn}}{R_2 + R_{tn}}$$

(b) To get expression being asked for, we need to use the recursive formula we have derived in part a. For $R_1 = R_2 = R$ the recursive formula becomes.

$$R_{t(n+1)} = R + \frac{RR_{tn}}{R + R_{tn}} \implies R_{t(n+1)} = R\left(1 + \frac{1}{1 + \frac{R}{R_{tn}}}\right)$$

Substituting the recursive formula into itself once, we get:

$$R_{t(n+1)} = R\left(1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{R}{R_{tn-1}}}}}\right)$$

It can be seen that if you continue to substitute the recursive formula into itself that you would get an equation of the form:

$$R_{tn} = R\left(1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\ldots}}}\right)$$

(c) Playing with infinities and producing some strange un-intuitive effects. If we have a resistor ladder with infinite levels, then removing one level would leave its length unchanged, hence:

$$R_{tn+1} = R_{tn} \implies R_{t(\infty)} = R + \frac{RR_{\infty}}{R + R_{t\infty}}$$

Using this fact the problem will get reduced to that of solving a second order polynomial.

$$R_{t\infty}^2 - R_{t\infty}R - R^2 = 0$$

Using the quadratic formula the solutions to this polynomial are:

$$R_{t\infty} = \frac{R \pm \sqrt{R^2 + 4R^2}}{2} = R(\frac{1 \pm \sqrt{5}}{2})$$

We currently have two solutions but one of them is non-physical/ $\sqrt{5} > 1$, hence if we were to take the negative solution then our resistance would be less than zero which can not occur for our set-up. Therefore our solution is:

$$R_{t\infty} = R(\frac{1 + \sqrt{5}}{2}).$$

We would also like to note that this coefficient of R has a special name. It's called the golden ratio.

Question 17

Imagine you are on a boat in a river holding a sphere, whose density exceeds that of water. You let the ball go and it sinks into the river. What happens to the water level; does it go up, down or stay the same? Justify your answer with mathematics.

Solution 17

Let the weight of the human and boat system be M and the mass of the sphere be m. Since the boat is floating on top of the river the buoyancy force must balance the weight of our entire system.

$$g\rho_f V = (M+m)g,$$

where ρ_f is the density of the fluid and V the volume of the displaced fluid. Hence the volume of the displaced fluid before we drop the sphere into the water is:

$$v_{before} = \frac{M+m}{\rho_f}$$

The volume of the displaced fluid after the sphere is dropped is given by the additive sum of the displaced fluids due to the sphere and boat system. Since the sphere sinks, the volume of the displaced fluid for the sphere is just the volume of the sphere. Hence after the sphere is dropped the total displaced volume is:

$$v_{after} = \frac{M}{\rho_f} + \frac{m}{\rho_s},$$

where ρ_s is the density of the sphere.

$$\frac{V_{after}}{V_{before}} = \frac{M}{M+m} + \frac{\rho_f}{\rho_s}\frac{m}{M+m}$$

This equation is valid for all magnitudes of m, hence we consider the case in the limit that m becomes really large ($m \gg M$).

$$\frac{M}{M+m} + \frac{\rho_f}{\rho_s}\frac{m}{M+m} \longrightarrow \frac{\rho_f}{\rho_s}$$

Since $\rho_s > \rho_f$, it follows that $V_{after} < V_{before}$. We can thus say that the water level would sink.

Question 18

Consider a test mass placed at a point equidistant and co-linear from the centers of two fixed planets. If this test mass is perturbed vertically from its equilibrium position, show that it undergoes simple harmonic motion.

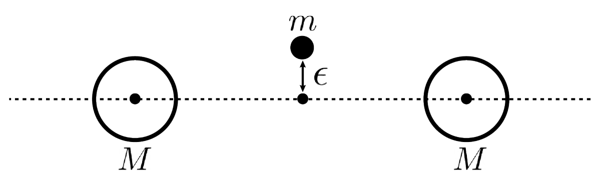

Figure 13: Mass system set-up.

Solution 18

Newton's law of gravitation is given by:

$$\vec{F} = -\frac{GMm}{r^2}\hat{r},$$

where M and m are masses of the objects interacting with each other, $\hat{r}$ is the unit vector connecting the two masses and r is the distance between them.

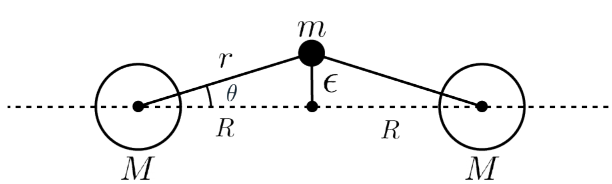

Figure 14: Solution schematic.

With reference to figure 14, taking the right to be the positive direction we can write the resultant force on the test mass as:

$$\sum \vec{F} = \frac{-GMm}{r^2}\begin{pmatrix} -cos(\theta) \\ sin(\theta) \end{pmatrix} + \frac{-GMm}{r^2}\begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix} = \frac{-GMm}{r^2}\begin{pmatrix} 0 \\ 2sin(\theta) \end{pmatrix}$$

Due to the x-component of the force being zero we drop the vector notation.

$$\sum F_y = \frac{-2GMm}{r^2} sin(\theta)$$

r^2 and $sin(\theta)$ can be written in terms of R and ϵ by utilizing the Pythagorean theorem and trigonometry.

$$r^2 = R^2 + \epsilon^2$$
$$sin(\theta) = \frac{\epsilon}{\sqrt{R^2 + \epsilon^2}}$$

To proceed further we will then apply a Taylor expansion. For a function $f(x)$, the Taylor expansions about a point $x = x_0$ is given by:

$$f(x) = f(x_0) + \sum_{j=0}^{\inf} f^{(j)}(x_0)(x - x_0)^j.$$

Expanding in ϵ/R about the origin $x_0 = 0$, r^{-2} and $sin(\theta)$ can be written:

$$r^{-2} = \frac{1}{R^2}(1 - \left(\frac{\epsilon}{R}\right)^2 + \mathcal{O}\left(\frac{\epsilon}{R}\right)^4)$$
$$sin(\theta) = \frac{\epsilon}{R} - \mathcal{O}\left(\frac{\epsilon}{R}\right)^2$$
$$\implies r^{-2}sin(\theta) = \frac{\epsilon}{R^3} + \mathcal{O}\left(\frac{\epsilon}{R}\right)^2$$

Since we are considering small perturbations about the equilibrium point, we can say that $\epsilon/R \ll 1$ and drop higher order terms $\mathcal{O}\left(\frac{\epsilon}{R}\right)^2$. Utilizing this result we see that the form of the resultant force reduces to Hooke's Law and the test mass undergoes simple harmonic motion.

$$\sum F_y = \frac{-2GMm}{R^3}\epsilon$$

Question 19

Find the effective resistance of the circuit between nodes a and b.

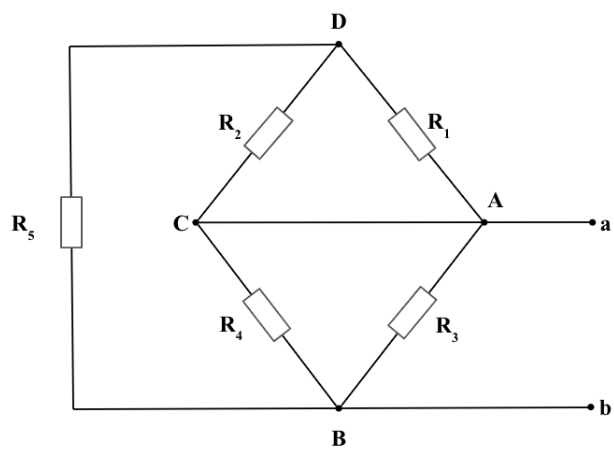

Figure 15: The circuit.

Solution 19

Note: the solution provided here is just one way to tackle circuits. Everyone has their own process and there are numerous ways to get to the same answer.

The first step is to realise that a current will definitely choose to flow through the wire section A to C as it effectively has no resistance. We can therefore treat A and C as the same node.

We will then proceed by considering the possible paths a current can flow through to get from A (or a) to B (or b). We will consider a series of OR and AND statements in the paths we take. ORs correspond to parallel elements and ANDs correspond to series. This will allow us to draw an equivalent (simplified) circuit.

$$A \longrightarrow D \longrightarrow B \qquad OR \qquad A \longrightarrow B$$

Seems obvious right? Next let's look at the resistors we need to encounter for each of these paths:

- $A \longrightarrow D \longrightarrow B$

  For $A \longrightarrow D$ we can go through R_1 OR R_2.

After that the current proceeds to the next step so we have an AND
For $D \longrightarrow B$ we can only go through R_5.

Note: $A = C$ so any path from A is the same as any path as C and vice versa.

- $A \longrightarrow B$

  We have two choices: R_3 OR R_4.

Now the initial node steps we defined tells us the overall "macro" branch structure of the new equivalent circuit. We had two possible paths for nodes corresponding to two branches. The resistor network of ANDs and ORs tells us the branch sub-structure. From the above we summarise:

- Branch 1: $(A \longrightarrow D \longrightarrow B)$

  1 parallel circuit with two resistors in series with one resistor.

- Branch 2: $(A \longrightarrow B)$

  1 parallel circuit with two resistors.

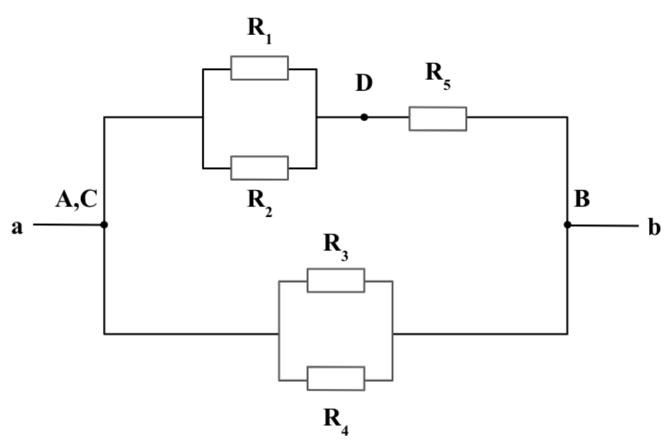

Figure 16: The equivalent circuit.

Finding the resistance is now a lot simpler!

$$\frac{1}{R_{eq}} = \frac{1}{\frac{R_1 R_2}{R_1+R_2} + R_5} + \frac{R_3 + R_4}{R_3 R_4}$$

We'll leave the inverting as an exercise for the reader.

Question 20

Show that the number of unique ways you can split n identical balls into k groups, is given by:

$$\binom{n+k-1}{k-1} \quad \text{or} \quad \binom{n+k-1}{n}$$

We assume that certain groups can also be empty and define the following notation:

$$\binom{p}{r} = \frac{p!}{(p-r)!r!}$$

Solution 20

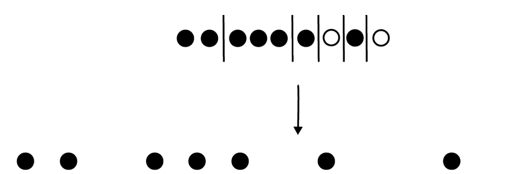

Figure 17: We depict how a arrange split group of balls are placed into baskets where the empty baskets that remain are where the partitions will be placed. The hollow circles depict a empty group. In the basket representation a empty group is represented by consecutive baskets containing partitions.

To tackle this problem we will consider the partitions that divide the balls into groups as objects too. For k groups there will be $k-1$ partitions.This brings our total number of objects from n to $n+k-1$.

If we imagine having $n+k-1$ baskets in a line, we can place n balls into n basket slots of our choosing. In the remaining baskets we can place the partitions. More explicitly we define these groups as follows:

- All baskets $(n+k-1)$ must have either a ball or a partition in it.
- Each basket can only contain one ball/ partition.
- An empty group is defined as a pair of adjacent baskets containing partitions.

After this procedure has been completed, we have achieved one of the possible configurations for the groups of balls (see figure 17 for an example).

$$\text{First ball : } n+k-1 \text{ baskets to choose from.}$$
$$\text{Second ball: } n+k-2 \text{ baskets to choose from.}$$
$$\vdots$$
$$(n-1)^{th} \text{ ball: } n+k-(n-1) \text{ baskets to choose from.}$$
$$n^{th} \text{ ball: } n+k-(n) \text{ baskets to choose from.}$$

Hence the total number of permutations for placing n balls into $n+k$ baskets is given by:

$$P = (n+k-1)(n+k-2)\ldots(n+k-(n-1))(n+k-n) = \frac{(n+k-1)!}{(k-1)!}$$

To complete a configuration for our placement of balls we still need to place our partitions. The total number of permutations of configurations P_c is the same as the number of permutations for placing n balls into $n+k-1$ baskets. This is because of the partitions being identical.

To see why, consider placing the balls into the baskets again. Since the partitions are identical no matter how you place them you will always get the same configuration. Hence, due to the number of ways to uniquely place partitions being one, it follows that $P_c = P$.

Any given configuration is invariant under the action of switching two balls since all balls are identical. For n balls there are $n!$ ways to switch them and keep a configuration invariant. Hence to get the number of unique ways to split n balls into k groups we need to divide P_c by $n!$ to account for over-counting.

$$\frac{P_c}{n!} = \frac{(n+k-1)!}{n!(k-1)!} = \frac{(n+k-1)!}{(n+k-1-(k-1))!(k-1)!} = \binom{n+k-1}{k-1}$$

Conversely you could have decided to place the partitions first and gone through the same procedure, which would result in you getting the following solution:

$$\binom{n+k-1}{n}$$

For anyone interested this is called a weak composition.

Question 21

The figure shows a system in which block A is used to push block B. All surfaces are subject to the influence of friction as the two blocks move. The coefficients of friction in the system are defined as follows:

1. Between the slope (at angle θ) and block A: μ_1
2. Between block A and block B: μ_2
3. Between block B and the horizontal ground: μ_3

Block A has mass $m_A kg$ and block B has mass $m_B kg$. What is the minimum magnitude of the downwards force P?

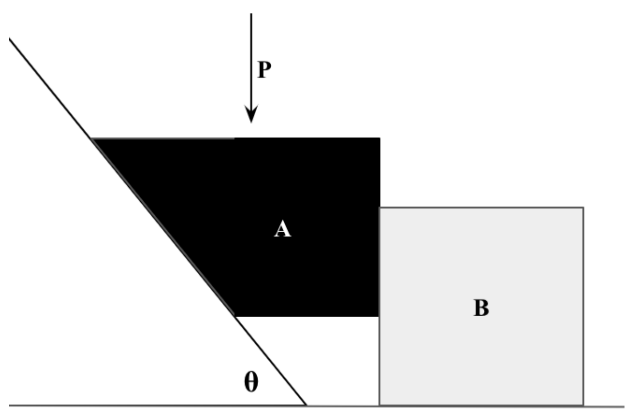

Figure 18: The system.

Solution 21

The minimum force that would need to be applied (P_{min}) would be the force for which the system is at the brink of moving. This means that up till this point, the whole system is in equilibrium and there should be no resultant forces in the x or y directions. We will exploit this fact in our calculations.

We start by drawing two free-body diagrams for the two blocks A and B, then consider their equilibrium independently.

Consider first **block B** (figure 19):

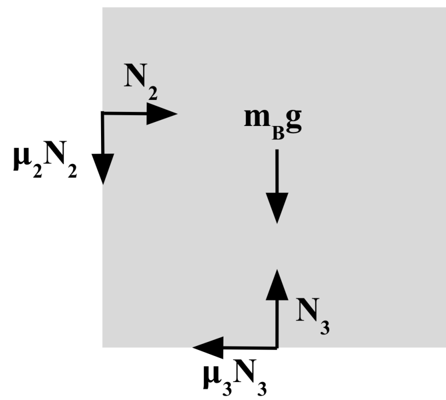

Figure 19: Free-body diagram of B.

1. X-Equilibrium Condition: $\sum F_x = 0$

$$N_2 - \mu_3 N_3 = 0 \implies N_3 = \frac{N_2}{\mu_3}$$

2. Y-Equilibrium Condition: $\sum F_y = 0$

$$N_3 - \mu_2 N_2 - m_B g = 0$$

Substituting the result from (1) into (2) we reach an expression for the reaction force between the two blocks:

$$\frac{N_2}{\mu_3} - \mu_2 N_2 - m_B g = 0$$

$$N_2 \left(\frac{1 - \mu_2 \mu_3}{\mu_3}\right) = m_B g$$

$$\implies N_2 = \frac{m_B g \mu_3}{1 - \mu_2 \mu_3}$$

We put this aside for now because we will need it later!

Now let us consider **block A** via the same method (figure 20):

1. X-Equilibrium Condition: $\sum F_x = 0$

$$-N_2 + N_1 sin\theta - \mu_1 N_1 cos\theta = 0$$

$$N_1(sin\theta - \mu_1 cos\theta) = N_2$$

$$\implies N_1 = \frac{N_2}{sin\theta - \mu_1 cos\theta}$$

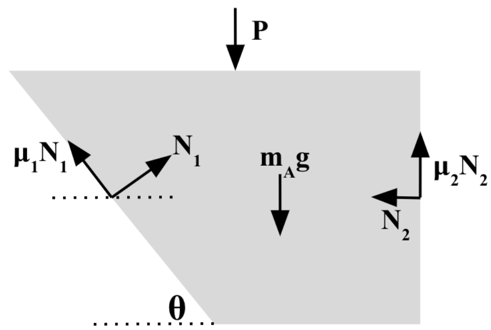

Figure 20: Free-body diagram of A.

2. Y-Equilibrium Condition: $\sum F_y = 0$

$$-P + \mu_2 N_2 - m_A g + N_1 cos\theta + \mu_1 N_1 sin\theta = 0$$

$$P = \mu_2 N_2 - m_A g + N_1(cos\theta + \mu_1 sin\theta)$$

Applying the result in (1) (expression for N_1 in terms of N_2) into (2), we get closer to an expression for P.

$$P = N_2(\mu_2 + \frac{(cos\theta + \mu_1 sin\theta)}{(sin\theta - \mu_1 cos\theta)}) - m_A g$$

Finally applying our expression for N_2 we can define P as a function of only the information we were given about the system!

$$P = \frac{m_B g \mu_3}{1 - \mu_2 \mu_3}(\mu_2 + \frac{(cos\theta + \mu_1 sin\theta)}{(sin\theta - \mu_1 cos\theta)}) - m_A g$$